Stochastic control, default risk and liquidity risk

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Quelques applications du contrôle stochastique aux risques de défaut et de liquidité

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INTRODUCTION GÉNÉRALE

Cette thèse se compose de trois parties indépendantes portant sur l'application du contrôle stochastique à la finance.

Dans la première partie, nous étudions la maximisation d'utilité de la richesse terminale dans un modèle avec défauts (ou sauts de type poissonnien) dans le cadre d'une information totale puis d'une information partielle. Nous nous intéressons aux fonctions d'utilité classiques : exponentielle, logarithmique et puissance. Cette étude est faite dans le cas de stratégies à valeurs dans un ensemble compact puis dans le cas non contraint. Le cas compact est résolu simplement grâce à un théorème de vérification. Dans le cas non contraint, grâce à des techniques de programmation dynamique, la fonction valeur associée à ce problème peut être caractérisée comme la solution d'une équation différentielle stochastique rétrograde (EDSR). Elle peut également être caractérisée comme la limite croissante (ou décroissante pour l'utilité exponentielle) d'une suite de solutions d'EDSR lipschitziennes. Ce résultat permet d'approcher numériquement la fonction valeur. En utilisant ces résultats, on obtient une caractérisation et une approximation du prix d'indifférence d'un actif contingent non duplicable.

Dans la deuxième partie, nous nous intéressons aux EDSR à sauts et plus particulièrement aux EDSR quadratiques. Celles-ci sont généralement utilisées en finance pour la résolution du problème de maximisation d'utilité de la richesse terminale en prenant pour fonction d'utilité la fonction exponentielle ou puissance. Nous utilisons la décomposition des processus à sauts liée au grossissement progressif de filtrations pour nous ramener à des EDSR browniennes entre les sauts. Cette méthode nous permet d'établir un théorème d'existence ainsi qu'un théorème d'unicité. En utilisant ces techniques de décomposition, nous donnons également une décomposition de la formule de Feynman-Kac pour les équations intégro-différentielles, celle-ci s'écrivant sous forme d'un système récursif d'équations aux dérivées partielles. Ces résultats sont appliqués à l'évaluation et à la couverture d'une option européenne dans un marché complet, et à la résolution du problème de maximisation d'utilité exponentielle de la richesse terminale dans le cas de stratégies à valeurs dans un ensemble compact.

La troisième partie est plus numérique et porte sur l'étude de la liquidation d'un portefeuille dans un modèle de risque de liquidité. On entend par liquidité la liquidité du marché, qui correspond à la possibilité pour un investisseur d'effectuer une transaction au prix affiché et pour un volume important sans affecter le cours du titre. Dans les modèles classiques, on fait l'hypothèse d'un marché financier parfaitement liquide, ce qui ne correspond guère à la réalité du marché. En effet, dans la plupart des cas, le marché est peu liquide et représente donc un risque pour les investisseurs concernés. On essaye d'expliquer le phénomène de risque de liquidité à l'aide de la théorie des erreurs. Ceci nous permet de modéliser la fourchette *bid-ask*. Ces résultats sont appliqués au problème de liquidation d'un portefeuille en temps discret et déterministe dans le modèle obtenu.

Dans la suite de cette introduction, nous allons exposer la problématique de chaque chapitre ainsi que les résultats importants obtenus.

0.1 Première partie : maximisation d'utilité dans un modèle avec défauts

Dans le contexte d'un marché incomplet, du fait de l'absence de stratégie de réplication, on va chercher à redéfinir la notion de stratégie optimale. Ceci est l'une des motivations conduisant à s'intéresser à des problèmes d'optimisation de la fonction d'utilité. Le problème particulier qui nous intéresse est celui de la maximisation de la fonction d'utilité de la richesse terminale d'un portefeuille. Nous regardons dans le premier chapitre la fonction d'utilité exponentielle, et dans le deuxième chapitre nous étudions les fonctions d'utilité logarithmique et puissance dans le cas d'une information totale et d'une information partielle. Rappelons qu'on parle d'une information partielle lorsque certaines des variables apparaissant dans le modèle ne sont pas observées.

Ce problème de maximisation d'utilité est très largement étudié dans la littérature. Dans l'article de référence de Merton [98], l'auteur examine un problème en temps continu de consommation-investissement d'un agent sur le marché. Il souhaite déterminer la proportion optimale de richesse que l'investisseur doit détenir pour chaque actif en fonction de son prix. En utilisant des techniques d'Hamilton-Jacobi-Bellman, l'auteur obtient une formule explicite de la fonction valeur associée au problème et la stratégie optimale correspondante.

Dans la littérature, on distingue deux approches pour résoudre ce problème de maximisation :

– l'approche duale, qui consiste à introduire le problème dual associé au problème d'op-

timisation, lorsque ce dernier est formulé de manière statique. On cite comme références, dans le cadre d'un marché complet, Karatzas, Lehoczky et Shreve [77] ou Cox et Huang [41], et dans le cadre d'un marché incomplet, Karatzas *et al.* [78], Kramkov et Schachermayer [84] ou Delbaen *et al.* [45] ;
- l'approche par contrôle stochastique, qui est basée sur le principe de la programmation dynamique (une formulation en est donnée dans El Karoui, Peng et Quenez [56]). On peut citer Jeanblanc et Pontier [71] dans le cadre d'un marché complet avec sauts, Rouge et El Karoui [117] dans le cas d'une filtration brownienne, Hu, Imkeller et Muller [67] dans le cas où les stratégies prennent leurs valeurs dans un ensemble fermé, Mania et Schweizer [97] pour des semimartingales générales ou Morlais [99] pour une filtration discontinue.

Concernant l'information partielle, la littérature est moins abondante. On peut citer dans le cas complet Detemple [47], Dothan et Feldman [48] ou Gennotte [63], qui utilisent le principe de la programmation dynamique dans un cadre gaussien, Lakner [86, 87], qui utilise l'approche martingale également dans un cadre gaussien, ou Karatzas et Zhao [81], qui utilisent l'approche duale dans un cadre bayesien. Dans un modèle incomplet, on peut citer Frey et Runggaldier [61] ou Lasry et Lions [88], qui étudient des problèmes de couverture, Pham et Quenez [110], qui traitent le cas de la volatilité stochastique, Platen et Runggaldier [113], qui s'intéressent à l'optimisation, Saas et Haussmann [120], qui regardent le cas markovien ou Callegaro, Di Masi et Runggaldier [32] et Roland [116], qui étudient le cas à sauts. On peut se référer à Runggaldier [119] pour un *survey* sur le filtrage.

Dans les deux chapitres de cette première partie, nous étudions un modèle avec défauts : on considère un marché financier incomplet constitué d'un actif sans risque dont le prix est supposé constant et égal à 1 et de n actifs risqués dont les prix à l'instant t sont notés $(S_t^i)_{1\leq i\leq n}$. On suppose que, sur le marché, il existe p instants de défaut (ou plus exactement de choc) que l'on note $(\tau_j)_{1\leq j\leq p}$. A chaque instant τ_j, $j\in\{1,\ldots,p\}$, les actifs risqués peuvent être discontinus. Dans le reste de cette introduction, on prend $n=p=1$ pour simplifier les notations. On suppose que le processus de prix suit la dynamique suivante :

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t),$$

avec W un mouvement brownien et N le processus correspondant à l'instant de défaut ($N_t = \mathbb{1}_{\tau\leq t}$). Nous notons $\mathbb{F} = \{\mathcal{F}_t, 0\leq t\leq T\}$ la filtration engendrée par (W,N), M la $\mathbb{F}$-martingale compensée de N et λ son $\mathbb{F}$-compensateur. Par la suite, nous considérons des stratégies π qui correspondent soit à la somme d'argent investie dans l'actif risqué pour le cas de la fonction d'utilité exponentielle, soit à la quantité d'actifs détenue pour le cas des fonctions d'utilité logarithmique et puissance. On note $X_T^{x,\pi}$ la richesse terminale associée à une richesse initiale x et à une stratégie π. Nous nous intéressons au problème de maximisation de l'espérance de la fonction d'utilité de la richesse terminale $X_T^{x,\pi}$:

$$V(x,\xi) = \sup_{\pi} \mathbb{E}\big[U\big(X_T^{x,\pi} + \xi\big)\big],$$

où ξ est un actif contingent non duplicable (ξ sera égal à 0 pour les fonctions d'utilité logarithmique et puissance). Une fois caractérisée la fonction valeur, nous déterminons le prix d'indifférence lorsque la fonction d'utilité est la fonction exponentielle. Celui-ci correspond à la somme à payer à l'instant initial pour recevoir la valeur associée à l'actif contingent à l'instant terminal. On le définit comme la somme p à retrancher à la richesse initiale x pour que le supremum de l'espérance de l'utilité de la richesse terminale soit le même entre un agent possédant l'actif contingent et un agent ne le possédant pas :

$$\sup_{\pi} \mathbb{E}\big[U\big(X_T^{x,\pi}\big)\big] = \sup_{\pi} \mathbb{E}\big[U\big(X_T^{x-p,\pi} + \xi\big)\big].$$

0.1.1 Maximisation de la fonction d'utilité exponentielle et prix d'indifférence dans un marché avec défaut

Dans le premier chapitre, nous nous intéressons au cas de la fonction d'utilité exponentielle $U(x) = -\exp(-\gamma x)$ où $\gamma > 0$ est une constante représentant l'aversion au risque de l'investisseur. Puisque l'égalité $V(x,\xi) = \exp(-\gamma x)V(0,\xi)$ est vérifiée, il est suffisant d'étudier le cas où la richesse initiale x est nulle. Pour simplifier les notations nous écrivons X_t^{π} à la place de $X_t^{0,\pi}$. Le gain réalisé entre t et s, correspondant à une stratégie π, est noté $X_s^{t,\pi} = \int_t^s \pi_u \frac{dS_u}{S_{u^-}}$. A chaque instant t, on définit la fonction valeur $J(t,\xi)$ par la variable aléatoire :

$$J(t,\xi) = \operatorname*{ess\,inf}_{\pi} \mathbb{E}\Big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\Big|\mathcal{F}_t\Big].$$

Nous étudions d'abord le cas où les stratégies π sont supposées à valeurs dans un *ensemble compact* C et les coefficients μ, σ, β et λ sont supposés bornés. En utilisant un *principe de vérification* (différent de celui de Hu *et al.* [67]) appliqué aux EDSR (dans l'esprit de celui d'El Karoui *et al.* [56]), on obtient facilement une caractérisation de la fonction valeur et de la stratégie optimale.

Théorème 0.1.1. *Soit (Y,Z,U) la solution dans $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ de l'EDSR suivante :*

$$\begin{cases} -dY_t &= \operatorname*{ess\,inf}_{\pi \in C} \Big\{\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2 Y_t - \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(Y_t + U_t)\Big\}dt \\ &\quad - Z_t dW_t - U_t dM_t, \\ Y_T &= \exp(-\gamma\xi). \end{cases} \tag{0.1.1}$$

Alors, $J(t,\xi) = Y_t$, $\mathbb{P}-p.s.$, pour tout $t \in [0,T]$. Il existe une unique stratégie optimale $\hat{\pi}$ à valeurs dans le compact C. Elle est caractérisée par le fait qu'elle est l'argument minimum du générateur de l'EDSR.

Notons qu'en faisant le changement de variable $y_t = \log(Y_t)$, on a que le processus y est solution d'une EDSR quadratique. On retrouve donc le résultat établi par Morlais [99] via des techniques d'EDSR quadratiques.

Dans le *cas sans contrainte*, il n'est pas possible d'utiliser un *théorème de vérification* comme pour le *cas compact*. Dans un premier temps, nous nous attardons sur le choix d'un ensemble approprié de stratégies admissibles. En effet, le principe de la programmation dynamique (PPD) n'est pas vérifié sur n'importe quel ensemble : des propriétés de recollement doivent être satisfaites, mais également des conditions d'intégrabilité car il s'agit d'un essentiel infimum (et non un essentiel supremum de variables aléatoires positives). Pour cela, on peut choisir par exemple l'ensemble $\mathcal{A}$ des stratégies π telles qu'il existe, pour chaque s, une constante $K_{s,\pi}$ telle que $X_t^{s,\pi} \geq -K_{s,\pi}$ pour tout $s \leq t \leq T$. Nous montrons que, sur cet ensemble, le PPD est satisfait : $(J(t,\xi))_{0\leq t\leq T}$ est le plus grand processus tel que $(\exp(-\gamma X_t^\pi)J(t,\xi))_{0\leq t\leq T}$ est une sous-martingale pour tout $\pi \in \mathcal{A}$ avec comme condition terminale $J(T,\xi) = \exp(-\gamma\xi)$.
A priori, on ne sait pas s'il existe une stratégie optimale sur l'ensemble $\mathcal{A}$, mais on peut tout de même caractériser la fonction valeur à l'aide d'une EDSR (sans aucune hypothèse de bornitude sur les coefficients).

Théorème 0.1.2. *Soit* (Y, Z, U, K) *la plus grande des sous-solutions dans* $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$ *de l'EDSR suivante :*

$$\begin{cases} -\,dY_t \;=\; \operatorname*{ess\,inf}_{\pi\in\mathcal{A}} \Big\{ \dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2 Y_t - \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(Y_t + U_t) \Big\} dt \\ \qquad\qquad -\, dK_t - Z_t dW_t - U_t dM_t, \\ Y_T \;=\; \exp(-\gamma\xi). \end{cases} \tag{0.1.2}$$

Alors, $J(t,\xi) = Y_t,\ \mathbb{P}-p.s.$*, pour tout* $t \in [0,T]$.

De plus, grâce à des techniques de contrôle, on montre (sans aucune hypothèse de bornitude sur les coefficients) que la fonction valeur $J(t,\xi)$ peut être approchée par une suite de processus $(J^k(t,\xi))_{k\in\mathbb{N}}$ où $J^k(t,\xi)$ est défini par :

$$J^k(t,\xi) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^k} \mathbb{E}\Big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\Big|\mathcal{F}_t\Big],$$

$\mathcal{A}^k$ étant l'ensemble des stratégies de $\mathcal{A}$ bornées par k.

Théorème 0.1.3. $\lim_{k\to\infty} \downarrow J^k(t,\xi) = J(t,\xi)$.

Dans le cas où les *coefficients sont supposés bornés*, grâce au théorème 0.1.1 appliqué à J^k, on obtient que $J^k(t,\xi)$ est la solution de l'EDSR (0.1.1) avec $C = [-k,k]$ laquelle est lipschitzienne. Ce résultat peut être utilisé pour approcher la fonction valeur grâce à des méthodes numériques. De plus, en utilisant un résultat de convergence de Morlais [99] établi grâce à des techniques d'EDSR quadratiques appliqué à $\log(J(t,\xi))$ et $\log(J^k(t,\xi))$, on en déduit que la suite de processus $(J^k(t,\xi))_{k\in\mathbb{N}}$ tend vers une solution de l'EDSR (0.1.2) avec $K=0$. Il s'en suit :

Corollaire 0.1.1. *Si les coefficients sont bornés, la fonction valeur $J(t,\xi)$ est la solution maximale de l'EDSR* (0.1.2) *($K=0$).*

Un paragraphe de ce chapitre est consacré à l'étude du cas où les coefficients sont non bornés mais satisfont une hypothèse d'*intégrabilité de type exponentiel.*

Nous pouvons alors caractériser le prix d'indifférence p d'un actif contingent non duplicable ξ à l'aide de solutions d'EDSR :

$$p=\frac{1}{\gamma}\ln\Big(\frac{J(0,0)}{J(0,\xi)}\Big),$$

et l'approcher par la suite $(p^k)_{k\in\mathbb{N}}$ définie par :

$$p^k=\frac{1}{\gamma}\ln\Big(\frac{J^k(0,0)}{J^k(0,\xi)}\Big),$$

car nous avons :

$$p=\lim_{k\to\infty}p^k.$$

Nous avons également généralisé ces résultats au cas de sauts de type poissonnien.

0.1.2 Optimisation de portefeuille dans un marché avec défaut sous information totale/partielle

Dans le deuxième chapitre, nous nous intéressons tout d'abord au cas d'une information totale et des fonctions d'utilité logarithmique $U(x)=\log(x)$ et puissance $U(x)=x^\gamma$ où $0<\gamma<1$ est une constante représentant l'aversion au risque de l'investisseur. Pour ces fonctions d'utilité, l'ensemble des stratégies admissibles $\mathcal{A}(x)$ est l'ensemble des stratégies telles que la richesse est toujours positive ; pour la fonction d'utilité logarithmique, on rajoutera des conditions d'intégrabilité afin de pouvoir résoudre le problème en adoptant une approche directe. Dans tout ce chapitre, nous supposons les coefficients bornés.

Le cas de la fonction d'utilité logarithmique peut être résolu directement :

Théorème 0.1.4. *La solution du problème d'optimisation est donnée par :*

$$V(x)=\log(x)+\mathbb{E}\Big[\int_0^T\Big(\hat{\pi}_t\mu_t-\frac{|\hat{\pi}_t\sigma_t|^2}{2}+\lambda_t\log(1+\hat{\pi}_t\beta_t)\Big)dt\Big],$$

avec $\hat{\pi}$ la stratégie optimale définie par :

$$\hat{\pi}_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{si } t < \tau \text{ et } \beta_t \neq 0, \\ \dfrac{\mu_t}{\sigma_t^2} \quad \text{si } t < \tau \text{ et } \beta_t = 0 \text{ ou } t \geq \tau. \end{cases}$$

On rappelle que, dans le cas d'un modèle sans défaut, la stratégie optimale est donnée par $\pi_t^0 = \mu_t/\sigma_t^2$. On remarque donc, dans le cas d'un modèle avec défaut, que la stratégie optimale peut s'écrire :

$$\hat{\pi}_t = \pi_t^0 - \epsilon_t,$$

où ϵ_t est un terme additionnel défini par :

$$\epsilon_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} + \dfrac{1}{2\beta_t} - \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{si } t < \tau \text{ et } \beta_t \neq 0, \\ 0 \quad \text{si } t < \tau \text{ et } \beta_t = 0 \text{ ou } t \geq \tau. \end{cases}$$

On remarque que, si le coefficient β est négatif (respectivement positif), *i.e.* le prix de l'actif diminue (resp. augmente) à l'instant de défaut, le terme additionnel est positif (resp. négatif), ce qui veut dire que l'agent doit investir une proportion de sa richesse plus petite (resp. grande) dans l'actif risqué que si le marché ne présentait pas de défaut.

Concernant la fonction d'utilité puissance, il suffit d'étudier uniquement le cas où la richesse initiale x est égale à 1 puisqu'on a l'égalité $V(x) = x^\gamma V(1)$. On note $\mathcal{A}$ à la place de $\mathcal{A}(1)$ et X_t^π à la place de $X_t^{1,\pi}$. Comme dans le chapitre 1, pour résoudre le problème de maximisation, nous rendons dynamique le problème initial en définissant pour chaque $t \in [0,T]$ la fonction valeur $J(t)$ par la variable aléatoire :

$$J(t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \mathbb{E}\big[\big(X_T^{t,\pi}\big)^\gamma \big| \mathcal{F}_t\big].$$

Mais, avant d'étudier le cas général, nous étudions le cas où l'ensemble admissible est l'ensemble des stratégies de $\mathcal{A}$ bornées par k, que l'on note $\mathcal{A}^k$. On note $J^k(t)$ la fonction valeur associée à cet ensemble. En appliquant un *principe de vérification*, on obtient une caractérisation de la fonction valeur $J^k(t)$ et de la stratégie optimale sur l'ensemble $\mathcal{A}^k$:

Théorème 0.1.5. *Soit (Y,Z,U) la solution dans $\mathcal{S}^2 \times L^2(W) \times L^2(M)$ de l'EDSR suivante :*

$$\begin{cases} -\,dY_t \;=\; -\,Z_t dW_t - U_t dM_t + \operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k} \Big\{ \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 Y_t \\ \qquad\qquad + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t + U_t) \Big\} dt, \\ Y_T \;=\; 1. \end{cases} \tag{0.1.3}$$

Alors, $J^k(t) = Y_t$, $\mathbb{P}-p.s.$, pour tout $t \in [0,T]$. Il existe une unique stratégie optimale $\hat{\pi} \in \mathcal{A}^k$. Elle est caractérisée par le fait qu'elle est l'argument maximum du générateur de l'EDSR.

Dans le cas général, le PPD est vérifié pour notre problème avec l'ensemble des stratégies admissibles classique $\mathcal{A}$: $(J(t))_{0\leq t\leq T}$ est le plus petit processus tel que $((X_t^\pi)^\gamma J(t))_{0\leq t\leq T}$ est une surmartingale pour tout $\pi \in \mathcal{A}$ avec comme condition terminale $J(T) = 1$.
Notons que comme il s'agit d'un essentiel supremum de fonctions positives, le PPD ne nécessite pas d'hypothèse d'intégrabilité comme c'était le cas dans le chapitre précédent.

Par la suite, nous faisons l'hypothèse classique :

$$J(0) < \infty.$$

Nous savons, d'après Kramkov et Schachermayer [84], que, sous cette hypothèse, il existe une stratégie optimale $\hat{\pi} \in \mathcal{A}$. De plus, sous cette même hypothèse, le critère d'optimalité est vérifié, c'est à dire :

Proposition 0.1.1. *Les assertions suivantes sont équivalentes :*

(i) *$\hat{\pi}$ est une stratégie optimale.*

(ii) *Le processus $((X_t^{\hat{\pi}})^\gamma J(t))_{0\leq t\leq T}$ est une martingale.*

Nous obtenons alors la caractérisation suivante de la fonction valeur :

Théorème 0.1.6. *Soit (Y,Z,U) la plus petite des solutions dans $L^{1,+} \times L^2_{loc}(W) \times L^1_{loc}(M)$ de l'EDSR suivante :*

$$\begin{cases} -\,dY_t &= -\,Z_t dW_t - U_t dM_t + \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{ \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 Y_t \\ &\qquad + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t + U_t)\Big\} dt, \\ Y_T &= 1. \end{cases} \tag{0.1.4}$$

Alors, $J(t) = Y_t$, $\mathbb{P}-p.s.$, pour tout $t \in [0,T]$. La stratégie optimale est un argument maximum du générateur de l'EDSR.

Notons que la démonstration de ce théorème est plus courte que dans le cas de la fonction d'utilité exponentielle car on a l'existence d'une stratégie optimale.

Grâce à des techniques de contrôle, on montre que la fonction valeur J peut être approchée par la suite de processus $(J^k(t))_{k\in\mathbb{N}}$:

$$J(t) = \lim_{k\to\infty} \uparrow J^k(t), \ \mathbb{P}-p.s.$$

Cela nous donne une méthode numérique pour approcher la fonction valeur $J(t)$ puisque les fonctions valeurs $(J^k(t))_{k\in\mathbb{N}}$ sont solutions d'EDSR lipschitziennes.

Nous supposons ensuite que l'agent sur le marché observe à chaque instant $t \in [0,T]$ uniquement le prix de l'actif S_t et le processus N_t. Par conséquent, les stratégies admissibles ne sont plus $\mathbb{F}$-prévisibles, mais $\mathbb{G}$-prévisibles, avec $\mathbb{G}$ la filtration engendrée par les prix observés et le temps de défaut. Nous supposons également que les coefficients σ et β sont markoviens :

$$\sigma_t = \sigma(t, S_{t^-}, t\wedge\tau) \text{ et } \beta_t = \beta(t, S_{t^-}, t\wedge\tau).$$

Afin d'appliquer les résultats obtenus, nous faisons tout d'abord une opération de filtrage. Pour cela on introduit les processus $\tilde{\mu}_t = \mathbb{E}[\mu_t|\mathcal{G}_t]$ et $\tilde{\lambda}_t = \mathbb{E}[\lambda_t|\mathcal{G}_t]$, ainsi que les processus $\bar{W}_t = W_t + \int_0^t(\mu_s - \tilde{\mu}_s)/\sigma_s ds$ et $\bar{M}_t = N_t - \int_0^t \tilde{\lambda}_s ds$. Nous avons alors :

- le processus $(\bar{W}_t)_{0\leq t\leq T}$ est un $\mathbb{G}$-mouvement brownien,
- le processus $(\bar{M}_t)_{0\leq t\leq T}$ est la $\mathbb{G}$-martingale compensée du processus N et $\tilde{\lambda}$ son $\mathbb{G}$-compensateur.

Ceci nous permet alors d'appliquer les résultats obtenus dans le cadre d'une information totale pour les fonctions d'utilité logarithmique, puissance et exponentielle, puisque le processus de prix suit la dynamique suivante :

$$dS_t = S_{t^-}(\tilde{\mu}_t dt + \sigma_t d\bar{W}_t + \beta_t dN_t).$$

On obtient alors, pour la fonction d'utilité logarithmique :

Théorème 0.1.7. *La fonction valeur est donnée par :*

$$V(x) = \log(x) + \mathbb{E}\Big[\int_0^T \Big(\hat{\pi}_t\tilde{\mu}_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \tilde{\lambda}_t\log(1+\hat{\pi}_t\beta_t)\Big)dt\Big],$$

avec $\hat{\pi}$ la stratégie optimale définie par :

$$\hat{\pi}_t = \begin{cases} \dfrac{\tilde{\mu}_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\tilde{\mu}_t\beta_t + \sigma_t^2)^2 + 4\tilde{\lambda}_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & si\ t<\tau\ et\ \beta_t \neq 0, \\ \dfrac{\tilde{\mu}_t}{\sigma_t^2} \quad si\ t<\tau\ et\ \beta_t = 0\ ou\ t\geq\tau. \end{cases}$$

Pour la fonction d'utilité puissance, nous avons :

Théorème 0.1.8. – *Soit $(\bar{Y},\bar{Z},\bar{U})$ la solution minimale dans $L^{1,+}\times L^2_{loc}(\bar{W})\times L^1_{loc}(\bar{M})$ de l'EDSR* (0.1.4) *avec $(W,\ M,\ \mu,\ \lambda)$ remplacé par $(\bar{W},\ \bar{M},\ \tilde{\mu},\ \tilde{\lambda})$, alors*

$$\bar{Y}_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \mathbb{E}\big[\big(X_T^{t,\pi}\big)^\gamma\big|\mathcal{G}_t\big],\ \mathbb{P}-p.s.$$

– *De plus, le processus* $\bar{Y}$ *est la limite croissante de la suite de processus* $(\bar{Y}^k)_{k\in\mathbb{N}}$, *où* $(\bar{Y}^k, \bar{Z}^k, \bar{U}^k)$ *est la solution dans* $\mathcal{S}^2 \times L^2(\bar{W}) \times L^2(\bar{M})$ *de l'EDSR* (0.1.3) *avec* $(W,\, M,\, \mu,\, \lambda)$ *remplacé par* $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$.

Pour la fonction d'utilité exponentielle, nous avons :

Théorème 0.1.9. – *Soit* $(\bar{Y}, \bar{Z}, \bar{U})$ *la solution maximale dans* $\mathcal{S}^{+,\infty} \times L^2(\bar{W}) \times L^2(\bar{M})$ *de l'EDSR* (0.1.2) *avec* $(W,\, M,\, \mu,\, \lambda)$ *remplacé par* $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$ *et* $K = 0$, *alors* $\bar{Y}_t = \bar{J}(t,\xi)$, $\mathbb{P} - p.s.$

– *De plus, le processus* $\bar{Y}$ *est la limite décroissante de la suite de processus* $(\bar{Y}^k)_{k\in\mathbb{N}}$, *où* $(\bar{Y}^k, \bar{Z}^k, \bar{U}^k)$ *est la solution dans* $\mathcal{S}^2 \times L^2(\bar{W}) \times L^2(\bar{M})$ *de l'EDSR* (0.1.1) *avec* $(W,\, M,\, \mu,\, \lambda)$ *remplacé par* $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$ *et* $C = [-k, k]$.

Ce théorème nous permet également de caractériser à l'aide de solutions d'EDSR le *prix d'indifférence* dans le cas d'une information partielle :

$$\bar{p} = \frac{1}{\gamma} \ln\Big(\frac{\bar{J}(0,0)}{\bar{J}(0,\xi)}\Big),$$

et également le *prix de l'information*, c'est-à-dire la différence de prix entre un agent non informé (qui a accès uniquement à l'information $\mathcal{G}_t$ à l'instant t) et un agent informé (qui a accès à l'information $\mathcal{F}_t$ à l'instant t) :

$$d = \bar{p} - p.$$

0.2 Grossissement progressif de filtrations et EDSR à sauts

Rappelons que les EDSR sont des équations de la forme suivante :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,\ 0 \le t \le T,$$

où W est un mouvement brownien sur un espace de probabilité $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ avec $\{\mathcal{F}_t,\, 0 \le t \le T\}$ la filtration engendrée par W, f est généralement appelé le générateur et ξ la condition terminale. Une solution est un couple (Y, Z) de processus $\mathbb{F}$-adapté vérifiant cette équation, Y et Z ont des propriétés d'intégrabilité dépendant des hypothèses sur le générateur f et sur la condition terminale ξ. Les EDSR ont été introduites par Bismut [18] pour le cas linéaire et par Pardoux et Peng [103] pour le cas général ; ils ont montré que, si le générateur f est lipschitzien et la condition terminale ξ est de carré intégrable, alors la solution existe et elle est unique. Depuis ce travail, la théorie des EDSR a connu un grand développement grâce notamment à ses applications en contrôle stochastique, en mathématiques financières et aux équations aux dérivées partielles. On peut citer en particulier le travail d'El Karoui, Peng et Quenez [56]. D'autres applications des EDSR pour le contrôle stochastique sont étudiées dans Hamadène et Lepeltier [64]. On peut également citer le livre d'El Karoui et Mazliak [55] pour les applications des EDSR. Il y a eu ensuite de nombreuses extensions portant sur le générateur. Kobylanski [83] a montré l'existence de solutions bornées pour un générateur à croissance quadratique en z, Lepeltier et San Martin [90] ont généralisé ces résultats au cas où f n'est pas à croissance linéaire en y. On peut également citer Briand et Hu [28] qui ont relaxé la condition "ξ bornée". Le cas à croissance quadratique en z trouve de nombreuses applications en finance, en particulier pour la résolution du problème de maximisation d'utilité exponentielle avec actif contingent. Ce problème a été largement étudié dans le cas continu, on peut citer sans être exhaustif Rouge et El Karoui [117], Sekine [124], Hu, Imkeller et Muller [67] et Mania et Schweizer [97].

Dans ce troisième chapitre, nous nous intéressons aux EDSR à sauts (EDSRS) du type

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(x)\mu(ds, dx), \ 0 \leq t \leq T,$$

où μ est une mesure aléatoire particulière représentant des temps de défaut aléatoires. Dans le cadre d'une mesure aléatoire classique, ces EDSRS ont été introduites par Tang et Li [126] qui prouvent l'existence et l'unicité d'une solution dans le cas où le générateur est lipschitzien. Puis Barles, Buckdahn et Pardoux [7] ont étudié le cas markovien. Royer [118] donne un théorème de comparaison pour les solutions de ces EDSRS. Le cas des EDSRS quadratiques est peu étudié. On peut citer Morlais [99] et El Karoui *et al.* [58] pour ce genre d'EDSR. Récemment, ces EDSRS ont été généralisées au risque de défaut, comme dans les chapitres un et deux de cette thèse. On peut citer le travail de Peng et Xu [109] qui donnent quelques applications des EDSRS au risque de défaut. Le cas quadratique est étudié dans Ankirchner *et al.* [3] pour résoudre le problème de maximisation d'utilité exponentielle dans un modèle avec un défaut, mais les auteurs font des hypothèses fortes sur le générateur.

Dans cette partie, nous utilisons des techniques de grossissement progressif de filtrations afin de faire un lien entre les EDSR browniennes et les EDSRS. Le grossissement de filtrations trouve ses origines dans Jeulin [74], Jeulin et Yor [75] et Jacod [69]. Depuis quelques années, ces travaux ont trouvé de nombreuses applications dans le risque de crédit puisqu'ils fournissent des outils puissants pour modéliser le risque de défaut. On peut trouver des applications, dans Bielecki, Jeanblanc et Rutkowski [14], Bielecki et Rutkowski [16] et Jiao et Pham [76] pour ne citer qu'eux. Dans la suite, nous utilisons principalement la décomposition des processus prévisibles et optionnels donnée dans Pham [111], laquelle est une généralisation de la décomposition de Jeulin [74].
Nous nous plaçons dans un espace de probabilité $(\Omega, \mathcal{G}, \mathbb{P})$ muni de la filtration $\{\mathcal{F}_t,\, 0 \leq t \leq T\}$ engendrée par W. Nous considérons une suite finie $(\tau_k, \zeta_k)_{1 \leq k \leq n}$ où :

– $(\tau_k)_{1\leq k\leq n}$ est une suite de variables aléatoires,
– $(\zeta_k)_{1\leq k\leq n}$ est une suite de marques aléatoires à valeurs dans un sous-ensemble borélien E de $\mathbb{R}^m$.

Nous notons μ la mesure aléatoire associée à la suite $(\tau_k, \zeta_k)_{1\leq k\leq n}$:

$$\mu([0,t]\times B) \ = \ \sum_{k=1}^{n} \mathbb{1}_{\{\tau_k\leq t,\, \zeta_k\in B\}},$$

Nous notons $\mathbb{G} = \{\mathcal{G}_t,\, 0\leq t\leq T\}$ la filtration engendrée par W, les temps de défaut $(\tau_k)_{1\leq k\leq n}$ et les marques $(\zeta_k)_{1\leq k\leq n}$. Nous remarquons dans un premier résultat qu'il est possible de prendre les temps de défaut ordonnés grâce aux marques. Par la suite nous noterons $\tau_{(k)}$ et $\zeta_{(k)}$ à la place de $(\tau_1,\ldots,\tau_k)$ et $(\zeta_1,\ldots,\zeta_k)$. D'après [111], nous obtenons les décompositions suivantes :

Lemme 0.2.1. – *Tout processus $\mathbb{G}$-prévisible $Y = (Y_t)_{0\leq t\leq T}$ admet une décomposition de la forme :*

$$Y_t = Y_t^0 \mathbb{1}_{t\leq\tau_1} + \sum_{k=1}^{n-1} Y_t^k(\tau_{(k)},\zeta_{(k)})\mathbb{1}_{\tau_k<t\leq\tau_{k+1}} + Y_t^n(\tau_{(n)},\zeta_{(n)})\mathbb{1}_{\tau_n<t},\ \ 0\leq t\leq T,$$

avec $Y^0\in\mathcal{P}_{\mathbb{F}}$, et $Y^k\in\mathcal{P}_{\mathbb{F}}^k(\Delta_k, E^k)$, pour tout $k=1,\ldots,n$.

– *Tout processus $\mathbb{G}$-optionnel $Y = (Y_t)_{0\leq t\leq T}$ admet une décomposition de la forme :*

$$Y_t = Y_t^0 \mathbb{1}_{t<\tau_1} + \sum_{k=1}^{n-1} Y_t^k(\tau_{(k)},\zeta_{(k)})\mathbb{1}_{\tau_k\leq t<\tau_{k+1}} + Y_t^n(\tau_{(n)},\zeta_{(n)})\mathbb{1}_{\tau_n\leq t},\ \ 0\leq t\leq T,$$

avec $Y^0\in\mathcal{O}_{\mathbb{F}}$, et $Y^k\in\mathcal{O}_{\mathbb{F}}^k(\Delta_k, E^k)$, pour tout $k=1,\ldots,n$.

Nous notons dans la suite de l'introduction Y_t^k et $Y_t^k(t,e)$ au lieu de $Y_t^k(\tau_{(k)},\zeta_{(k)})$ et $Y_t^k(\tau_{(k-1)},t,\zeta_{(k-1)},e)$. En utilisant les décompositions de ξ et de f, nous obtenons un résultat d'existence de solution aux EDSRS de la forme suivante :

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds - \int_t^T Z_s dW_s - \int_t^T\int_E U_s(e)\mu(de,ds)\,. \tag{0.2.5}$$

Théorème 0.2.10. *Supposons que, pour tout $(\theta,e)\in\Delta_n\times E^n$, l'EDSR :*

$$Y_t^n(\theta,e) = \xi^n(\theta,e) + \int_t^T f^n(s,Y_s^n(\theta,e),Z_s^n(\theta,e),0,\theta,e)ds - \int_t^T Z_s^n(\theta,e)dW_s$$

admette une solution $(Y^n(\theta,e),Z^n(\theta,e))\in\mathcal{S}_{\mathbb{F}}^\infty\times L_{\mathbb{F}}^2(W)$, et que, pour chaque $k=0,\ldots,n-1$, l'EDSR :

$$\begin{aligned} Y_t^k(\theta_{(k)},e_{(k)}) \ &= \ \xi^k(\theta_{(k)},e_{(k)}) + \int_t^T f^k(s,Y_s^k(\theta_{(k)},e_{(k)}),Z_s^k(\theta_{(k)},e_{(k)}), \\ &\qquad Y_s^{k+1}(\theta_{(k)},s,e_{(k)},.) - Y_s^k(\theta_{(k)},e_{(k)}))ds - \int_t^T Z_s^k(\theta_{(k)},e_{(k)})dW_s \end{aligned}$$

admette une solution $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)})) \in \mathcal{S}^\infty_\mathbb{F} \times L^2_\mathbb{F}(W)$. *Supposons de plus que chaque* Y^k *(resp.* Z^k*) est* $\mathcal{O}_\mathbb{F} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$*-mesurable (resp.* $\mathcal{P}_\mathbb{F} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$*-mesurable).*

Si toutes ces solutions satisfont :

$$\sup_{(k,\theta,e)\in\{0,\dots,n\}\times\Delta_n\times E^n} ||Y^k(\theta_{(k)}, e_{(k)})||_{\mathcal{S}^\infty_\mathbb{F}} < \infty$$

et

$$\int_{\Delta_n\times E^n} \mathbb{E}\Big[\int_0^{\theta_1\wedge T} |Z^0_s|^2 ds + \sum_{k=1}^n \int_{\theta_k\wedge T}^{\theta_{k+1}\wedge T} |Z^k_s(\theta_{(k)}, e_{(k)})|^2 ds\Big] \gamma(\theta, e) d\theta\, \eta(de) < \infty ,$$

alors l'EDSR (0.2.5) admet une solution $(Y, Z, U) \in \mathcal{S}^\infty_\mathbb{G} \times L^2_\mathbb{G}(W) \times L^2(\mu)$ *donnée par :*

$$\begin{cases} Y_t = & Y^0_t \mathbb{1}_{t<\tau_1} + \displaystyle\sum_{k=1}^n Y^k_t(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \le t < \tau_{k+1}} , \\ Z_t = & Z^0_t \mathbb{1}_{t\le\tau_1} + \displaystyle\sum_{k=1}^n Z^k_t(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k < t \le \tau_{k+1}} , \\ U_t(.) = & U^0_t(.) \mathbb{1}_{t\le\tau_1} + \displaystyle\sum_{k=1}^{n-1} U^k_t(\tau_{(k)}, \zeta_{(k)}, .) \mathbb{1}_{\tau_k < t \le \tau_{k+1}} , \end{cases}$$

avec $U^0_t(.) = Y^1_t(t, .) - Y^0_t$ *et* $U^k_t(\tau_{(k)}, \zeta_{(k)}, .) = Y^{k+1}_t(\tau_{(k)}, t, \zeta_{(k)}, .) - Y^k_t(\tau_{(k)}, \zeta_{(k)})$ *pour chaque* $k = 1, \dots, n-1$.

Nous donnons des exemples explicites pour lesquels le théorème précédent s'appliquent.

Corollaire 0.2.2. *Supposons que la variable aléatoire* ξ *est bornée. Supposons également que le générateur* $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E \to \mathbb{R}$ *satisfait une des deux conditions suivantes :*

(i) f *est déterministe et lipschitzien : il existe une constante* C *telle que*

$$|f(t, y, z, u(e) - y)_{e\in E} - f(t, y', z', u(e) - y')_{e\in E}| \quad \le \quad C(|y - y'| + |z - z'|) ,$$

pour tout $(t, y, y', z, z', u) \in [0, T] \times [\mathbb{R}]^2 \times [\mathbb{R}^d]^2 \times \mathbb{R}^E$,

(ii) f *est quadratique en* z *: il existe une constante* C *telle que*

$$|f(t, y, z, u)| \quad \le \quad C(1 + |z|^2) ,$$

pour tout $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$.

Alors, l'EDSR (0.2.5) admet une solution dans $\mathcal{S}_{\mathbb{G}}^\infty \times L^2_{\mathbb{G}}(W) \times L^2(\mu)$.

Cette technique de décomposition des EDSRS nous permet également d'obtenir un théorème d'unicité, mais pour cela nous devons ajouter une hypothèse sur les temps d'arrêt $(\tau_k)_{1\leq k\leq n}$:

Hypothèse 0.2.1. Les temps d'arrêt $(\tau_k)_{1\leq k\leq n}$ sont inaccessibles dans la filtration $\mathbb{G}$.

Nous pouvons alors établir un théorème de comparaison pour les EDSRS. Soit deux EDSRS $(\underline{f},\underline{\xi})$ et $(\bar{f},\bar{\xi})$, et $(\underline{Y},\underline{Z},\underline{U})$ et $(\bar{Y},\bar{Z},\bar{U})$ leurs solutions respectives dans $\mathcal{S}_{\mathbb{G}}^\infty \times L^2_{\mathbb{G}}(W)\times L^2(\mu)$. Nous considérons les décompositions $(\underline{\xi}^k)_{0\leq k\leq n}$ (resp. $(\bar{\xi}^k)_{0\leq k\leq n}$, $(\underline{f}^k)_{0\leq k\leq n}$, $(\bar{f}^k)_{0\leq k\leq n}$, $(\underline{Y}^k)_{0\leq k\leq n}$, $(\bar{Y}^k)_{0\leq k\leq n}$, $(\underline{Z}^k)_{0\leq k\leq n}$, $(\bar{Z}^k)_{0\leq k\leq n}$, $(\underline{U}^k)_{0\leq k\leq n}$, $(\bar{U}^k)_{0\leq k\leq n}$) de $\underline{\xi}$ (resp. $\bar{\xi}$, $\underline{f}$, $\bar{f}$, $\underline{Y}$, $\bar{Y}$, $\underline{Z}$, $\bar{Z}$, $\underline{U}$, $\bar{U}$) (pour simplifier les notations, nous ne notons pas la dépendance en $(\theta_{(k)},e_{(k)})$). Et pour simplifier les notations, nous écrivons

- $\underline{F}^n(t,y,z)$ et $\bar{F}^n(t,y,z)$ à la place de $\underline{f}^n(t,y,z,0)$ et $\bar{f}^n(t,y,z,0)$,
- $\underline{F}^k(t,y,z)$ et $\bar{F}^k(t,y,z)$ à la place de $\underline{f}^k(t,y,z,Y_t^{k+1}(t,.)-y)$ et $\bar{f}^k(t,y,z,\bar{Y}_t^{k+1}(t,.)-y)$ pour chaque $k=0,\ldots,n-1$.

Nous obtenons le théorème de comparaison suivant :

Théorème 0.2.11. *Supposons que* $\underline{\xi}\leq\bar{\xi}$, $\mathbb{P}$-*p.s. Si pour chaque* $k=0,\ldots,n$

$$\underline{F}^k(t,y,z) \;\leq\; \bar{F}^k(t,y,z), \quad \forall(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d, \quad \mathbb{P}-p.s.\,,$$

et que l'un des générateurs $\bar{F}^k$ *ou* $\underline{F}^k$ *satisfait un théorème de comparaison pour les EDSR browniennes. Alors, si* $\bar{U}_t=\underline{U}_t=0$ *pour* $t>\tau_n$, *on a*

$$\underline{Y}_t \;\leq\; \bar{Y}_t\,, \quad \forall\, t\in[0,T]\,, \quad \mathbb{P}-p.s.$$

En particulier, pour les EDSRS quadratiques satisfaisant l'hypothèse suivante :

i) il existe une constante C telle que

$$\begin{cases} |f(t,y,z,u)| & \leq & C(1+|z|^2)\,, \\ \Big|\partial_z f(t,y,z,u)\Big| & \leq & C(1+|z|)\,, \end{cases} \tag{0.2.6}$$

pour tout $(t,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^E$, $\mathbb{P}$-p.s.,

ii) pour tout $\varepsilon>0$, il existe une constante C_ε telle que

$$\partial_y f(t,y,z,(u(e)-y)_{e\in E}) \;\leq\; C_\varepsilon+\epsilon|z|^2\,,$$

pour tout $(t,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^E$, $\mathbb{P}$-p.s.

Théorème 0.2.12. *Sous cette dernière hypothèse, l'EDSR (0.2.5) admet au plus une solution.*

Cette technique de décomposition nous permet également de considérer les équations intégro-différentielles de la forme :

$$\begin{cases} -\partial_t u(t,x) - \mathcal{L}u(t,x) - h(x, u(t,x), \sigma Du(t,x), \int_E (u(t, x+\beta(x,e)) - u(t,x))\gamma(x,e)\lambda(de) = 0 \\ \text{pour } (t,x) \in [0,T] \times \mathbb{R}^d \text{ et} \\ u(T,.) = g(.), \end{cases}$$

où $\mathcal{L}$ est l'opérateur local du second ordre :

$$\mathcal{L}u(t,x) \;=\; b(x)Du(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(x)D^2u(t,x)).$$

Nous obtenons alors le résultat suivant :

Théorème 0.2.13. *Soit v l'unique solution de l'équation intégro-différentielle précédente. Alors, nous avons :*

$$v(t,x) \;=\; Y_t^{0,t,x},$$

où la famille $(Y^{k,t,x}(\theta_{(k)}, e_{(k)}))_{0\leq k\leq n}$ est définie de manière récursive et chaque $Y^{k,t,x}(\theta_{(k)}, e_{(k)})$ est solution d'une EDSR brownienne.
De plus, nous avons la décomposition suivante de v :

$$v(t,x) \;=\; v_0(t,x),$$

où la famille $(v_k(., \theta_{(k)}, e_{(k)}))_{0\leq k\leq n}$ est définie de manière récursive et chaque $v_k(., \theta_{(k)}, e_{(k)})$ est solution d'une équation aux dérivées partielles.

Nous donnons deux exemples d'utilisation d'EDSRS :
- évaluation et couverture d'une option européenne dans un marché complet,
- détermination du prix d'indifférence d'un actif contingent non duplicable dans un marché incomplet.

On considère un marché financier constitué d'un actif sans risque S^0 et de deux actifs risqués S^1 et S^2. On suppose que sur ce marché, il existe un temps de défaut τ. L'actif sans risque suit l'équation :

$$dS_t^0 = r_t S_t^0 dt,$$

et les actifs risqués suivent l'équation :

$$\begin{cases} dS_t^1 = S_{t^-}^1(\mu_t dt + \sigma_t dW_t + \beta dM_t)\,, \\ dS_t^2 = S_t^2(\bar{\mu}_t dt + \bar{\sigma}_t dW_t)\,. \end{cases}$$

On considère que chaque coefficient a une valeur constante avant l'instant de défaut τ et une valeur constante après l'instant de défaut. L'évaluation d'une option européenne ξ revient à résoudre l'EDSRS suivante :

$$\begin{cases} -dY_t = \Big[\frac{r_t - \bar{\mu}_t}{\bar{\sigma}_t} Z_t + \Big(\frac{r_t - \mu_t}{\beta} + \lambda_t - \frac{\sigma_t(r_t - \bar{\mu}_t)}{\beta\bar{\sigma}_t}\Big) U_t - r_t Y_t\Big] dt - Z_t dW_t - U_t dN_t\,, \\ \quad Y_T = \xi\,. \end{cases}$$

En utilisant les techniques précédentes on donne une solution explicite du prix de cette option.

Nous considérons un marché constitué d'un actif sans risque constant et égal à 1, et d'un actif risqué S. On suppose que le processus de prix suit la dynamique suivante :

$$dS_t = S_{t^-}\Big(\mu_t dt + \sigma_t dW_t + \int_E \beta_t(e)\mu(de, dt)\Big),$$

où l'on suppose, de plus, que les coefficients sont uniformément bornés et que $\beta_t(e) > -1$. Une stratégie $\pi = (\pi_t)_{0\leq t\leq T}$ correspond à la somme d'argent investie dans l'actif risqué à la date t et on note $X_t^{x,\pi}$ la richesse à l'instant t associée à une stratégie π et une richesse initiale x. Nous cherchons à résoudre le problème de maximisation d'utilité exponentielle :

$$V(x) = \sup_{\pi\in C} \mathbb{E}\big[-\exp\big(-\alpha\big(X_T^{x,\pi} - B\big)\big)\big],$$

où B est un actif contingent borné, et C un ensemble compact.
Pour résoudre ce problème, nous utilisons un théorème de vérification qui permet alors de dire que :

$$V(x) = -\exp(-\alpha(x - Y_0))\,,$$

avec Y_0 la valeur initiale de la solution de l'EDSRS suivante :

$$Y_t = B + \int_t^T f(s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds),$$

où

$$f(t, z, u) = \inf_{\pi\in C}\Big\{\frac{\alpha}{2}\big|\pi_t\sigma_t - (z + \frac{\theta_t}{\alpha})\big|^2 + \int_E \frac{\exp(-\alpha(\pi_t\beta_t(e) - u(e))) - 1}{\alpha} n(de)\Big\} - \theta_t z - \frac{|\theta_t|^2}{2\alpha}.$$

Grâce aux techniques précédentes, nous prouvons que cette équation admet une unique solution, ce qui permet de caractériser la fonction valeur ainsi que la stratégie optimale, celle-ci étant définie comme l'argument minimum du générateur f. On peut alors déterminer le prix d'indifférence comme dans le chapitre un.

0.3 Modélisation du *spread bid-ask* : une approche perturbative

Généralement, dans les modèles classiques en mathématiques financières, les auteurs considèrent une parfaite élasticité des actifs, en supposant que les transactions n'ont aucun impact sur le prix de l'actif. Cependant, la littérature sur la microstructure du marché a montré théoriquement et empiriquement que les grosses transactions influencent significativement le prix de l'actif sous-jacent, démontrant ainsi l'existence du risque de liquidité. Par conséquent, comprendre le fonctionnement des marchés financiers est un enjeu fondamental pour les praticiens de la finance. Une question importante que se posent les agents sur le marché concerne la façon de liquider un portefeuille de N actifs, avec N relativement important. En effet un dilemme se pose : soit l'agent décide de tout vendre en une seule opération, auquel cas il est soumis à des coûts élevés dus à l'épuisement du carnet d'ordre, soit il vend en plusieurs opérations espacées d'un certain temps, mais, dans ce cas, l'agent est soumis aux variations du marché. Dans cette troisième partie, nous essayons d'expliquer le phénomène de liquidité, en utilisant la théorie des erreurs, comme une propriété intrinsèque du marché et nous étudions un problème de liquidation optimale d'un portefeuille en temps discret et déterministe.

On trouve dans la littérature trois approches pour modéliser le risque de liquidité. La première approche consiste à utiliser des fonctions d'impact pour modéliser la dépendance du prix d'un actif en fonction de la stratégie de *trading*. L'impact de la stratégie de *trading* sur la dynamique du prix peut être permanente, par exemple pour de gros investisseurs — on peut citer sans être exhaustif Frey [60], Platen et Schweizer [112] et He et Mamaysky [65] — ou temporaire pour de petits investisseurs — on peut citer Cetin, Jarrow et Protter [34], Cetin et Rogers [35] et Cetin, Soner et Touzi [36]. La seconde approche consiste à considérer la structure du marché et de modéliser le carnet d'ordre (voir par exemple Alfonsi, Schied et Schulz [1] et Cont, Stoikov et Talreja [37]). La troisième approche consiste non pas à modéliser le carnet d'ordre, mais uniquement la fourchette *Bid-Ask* ; généralement la modélisation de la fourchette *Bid-Ask* est associée à des fonctions d'impact (on peut citer Kharroubi et Pham [82] et Schied et Schoneborn [122]).

Dans cette troisième partie, nous essayons d'expliquer le risque de liquidité de manière différente en utilisant la théorie des erreurs développée par Bouleau [25, 26, 27] et ses travaux avec Hirsch sur les formes de Dirichlet [24], ce qui nous permet d'expliquer l'existence d'une fourchette *Bid-Ask* comme une propriété inhérente au marché. Une fois la modélisation de la fourchette *Bid-Ask* réalisée, comme dans Bertsimas et Lo [11], Almgren et Chriss [2], Obizhaeva et Wang [101] et Alfonsi *et al.* [1], nous étudions un problème de liquidation optimale d'un portefeuille en temps discret et déterministe. Afin de résoudre complètement ce problème, nous ne prenons pas uniquement en compte la fourchette *Bid-Ask*, mais également la profondeur dans le carnet d'ordre en rajoutant une fonction d'impact.

Nous considérons un marché financier comportant un actif risqué de processus de prix X suivant l'équation différentielle stochastique (EDS) :

$$dX_t = rX_t dt + \sigma(t, X_t, \omega) X_t dW_t,$$

mais on suppose que cet actif n'est pas échangeable (comme le CAC 40 par exemple). Par contre, il existe sur le marché un actif échangeable qui le réplique (un *tracker* par exemple) dont le processus de prix S suit l'EDS suivante :

$$dS_t = rS_t dt + \sigma(t, S_t, \omega) S_t dB_t,$$

où B est un mouvement brownien, qui n'est pas tout à fait égal à W à cause d'une incertitude :

$$B_t = \sqrt{e^{-\epsilon}} W_t + \sqrt{1 - e^{-\epsilon}} \hat{W}_t,$$

avec ϵ un petit paramètre et $\hat{W}$ un mouvement brownien indépendant de W et non observable. La théorie des erreurs nous permet de savoir comment l'incertitude sur le mouvement brownien B se répercute sur le processus de prix S. Pour chaque réalisation $\bar{\omega}$ du processus X au temps t, $S_t(\bar{\omega})$ est une variable aléatoire décrite par :

$$S_t(\bar{\omega}, \hat{\omega}) = X_t(\bar{\omega}) + \epsilon \mathcal{A}[S_t](\bar{\omega}) + \sqrt{\epsilon \Gamma[S_t](\bar{\omega})} \tilde{\mathcal{N}}(\hat{\omega}),$$

où $\tilde{\mathcal{N}}$ est une variable aléatoire gaussienne centrée réduite indépendante de W, et $\Gamma[S_t]$ et $A[S_t]$ sont donnés par :

$$\begin{cases} \Gamma[S_t] = \theta M_t^2 \displaystyle\int_0^t \frac{X_s^2 \sigma^2(s, X_s, \omega)}{M_s^2} ds + \Gamma[S_0] M_t^2, \\ \mathcal{A}[S_t] = M_t \displaystyle\int_0^t \frac{\eta(s, X_s, \omega)\Gamma[S_s] - \theta X_s \sigma(s, X_s, \omega)}{2M_s} \big[dW_s - \zeta(s, X_s, \omega) ds \big], \\ \quad M_t = \mathcal{E}\Big\{ \displaystyle\int_0^t \zeta(s, X_s, \omega) dW_s + rt \Big\}, \end{cases}$$

où $\mathcal{E}$ est l'exponentielle de Doleans-Dade.

Nous considérons que, sur le marché, il existe plusieurs agents ; ils sont tous informés sur l'évolution du prix du *benchmark*, mais n'ont aucune information sur la perturbation engendrée par $\hat{W}$. Nous supposons que tous les agents sont averses aux risques et peuvent estimer la distribution du prix S à tout instant t avec la formule de S_t. Parmi tous les agents, il en existe un qui a une aversion au risque minimale par rapport aux autres. Cet agent accepte d'acheter l'actif à un prix S_t^B plus élevé que les prix proposés par les autres agents, donc le prix proposé par cet agent est le prix *Bid* et est noté S_t^B. Ce prix est caractérisé uniquement par la loi de S_t et l'aversion au risque de cet agent. On définit de la même manière le prix *Ask* noté S_t^A. Nous supposons, par la suite, que l'agent qui a l'aversion au risque minimale est toujours le même, et qu'il propose le meilleur prix d'achat et de vente. Nous définissons les prix *Bid* et *Ask* de la manière suivante ($\chi_A + \chi_B < 1$) :

$$\begin{cases} S_t^B = X_t + \epsilon\mathcal{A}[S_t] + \sqrt{\epsilon\Gamma[S_t]}\tilde{\mathcal{N}}^{-1}(\chi_B), \\ S_t^A = X_t + \epsilon\mathcal{A}[S_t] + \sqrt{\epsilon\Gamma[S_t]}\tilde{\mathcal{N}}^{-1}(1-\chi_A). \end{cases}$$

Par la suite, on suppose que $\chi_A = \chi_B = \chi$. Pour déterminer la fourchette *Bid-Ask*, il nous reste à choisir la dynamique de l'aversion au risque de l'agent ; pour cela, nous avons choisi de prendre $\tilde{\mathcal{N}}^{-1}(\chi) = \exp(Y_t)$ avec Y un processus d'Ornstein-Uhlenbeck. Ce modèle a été choisi en particulier parce qu'il possède les propriétés suivantes :

- le prix *Ask* est toujours plus grand que le prix *Bid*,
- tous les termes excepté X ont une forme explicite,
- le prix *Mid* est différent du prix du *benchmark*, ce qui explique le biais systémique,
- si le prix du *benchmark* est stable, la fourchette *Bid-Ask* a un comportement de retour à la moyenne.

Maintenant que nous avons défini les prix *Bid* et *Ask*, nous nous intéressons au problème de liquidation d'un portefeuille. Nous considérons un agent possédant N actifs dont il souhaite se débarrasser, mais, pour cela, il ne peut vendre qu'à des dates déterminées $t_1, \dots, t_n$. On dit qu'une stratégie $\pi = (\pi_1, \dots, \pi_n)$ est admissible si π est $(\mathcal{F}_{t_i})_{1 \leq i \leq n}$-adapté avec $\mathcal{F}$ la filtration engendrée par le prix X du *benchmark*, $0 \leq \pi_i \leq N$ et $\sum_{i=1}^n \pi_i = N$. Nous supposons que, lorsque l'agent vend x actifs à l'instant t, le prix moyen auquel il vend ses actifs est égal à $\bar{S}_t^B(x) = g(x)S_t^B$, avec g une fonction vérifiant certaines hypothèses. L'agent cherche alors à maximiser l'espérance de ses gains futurs

$$\mathbb{E}\Big[\sum_{i=1}^n e^{-\rho t_i}\pi_i \bar{S}_{t_i}^B(\pi_i)\Big].$$

Pour résoudre ce problème, nous définissons à chaque instant t_i et pour chaque p, nombre d'actifs restant à vendre à l'instant t_i, l'ensemble des stratégies admissibles $\mathcal{A}(t_i, p)$ par :

$$\mathcal{A}(t_i, p) = \Big\{\pi = \big\{\pi_i, \dots, \pi_n\big\},\ \pi_j \geq 0\ \forall j \in \{i, \dots, n\} \text{ et } \sum_{j=i}^n \pi_j = p\Big\}.$$

Pour un état z de la variable $Z_{t_i} = (X_{t_i}, A[S_{t_i}], \Gamma[S_{t_i}], Y_{t_i})$ et un état p de la variable P_{t_i}, représentant le nombre d'actifs restant à vendre à l'instant t_i, on définit la fonction gain pour chaque stratégie $\pi \in \mathcal{A}(t_i, p)$ par :

$$J(i, z, p, \pi) = \mathbb{E}\Big[\sum_{j=i}^n e^{-\rho(t_j - t_i)}\pi_j S_{t_j}^B g(\pi_j)\Big],$$

et nous définissons également la fonction valeur par :

$$v(i, z, p) = \sup_{\pi \in \mathcal{A}(t_i, p)} (J(i, z, p, \pi)).$$

Nous disons, pour un état (i, z, p), que la stratégie $\hat{\pi} \in \mathcal{A}(t_i, p)$ est optimale si :

$$v(i, z, p) = J(i, z, p, \hat{\pi}).$$

En utilisant le principe de la programmation dynamique, nous prouvons qu'il existe une unique stratégie optimale à notre problème de liquidation et celle-ci est donnée par l'argument maximum de :

$$\begin{cases} v(i, z, p) = \operatorname*{ess\,sup}_{0 \leq \pi_i \leq p} \Big\{ \pi_i s_i^B g(\pi_i) + \mathbb{E}\Big[e^{-\rho(t_{i+1} - t_i)} v\big(i+1, Z_{i+1}^{i,z}, p - \pi_i\big) \Big| \mathcal{F}_{t_i} \Big] \Big\}, \\ v(n,\, z,\, p) = p s_n^B g(p). \end{cases}$$

Puis, nous étudions numériquement le cas des modèles Black-Scholes et *constant elasticity of variance* (CEV). Dans le cas Black-Scholes, on voit que le nombre d'actifs à vendre à chaque date est indépendant de la valeur du sous-jacent (on retrouve le résultat d'Alfonsi *et al.* [1]), il est décroissant par rapport à la valeur de la fourchette *Bid-Ask* et est croissant par rapport au nombre d'actifs restant à vendre. Dans le cas CEV, on voit que le nombre d'actifs à vendre à chaque date dépend de la valeur du sous-jacent, il est décroissant par rapport à la valeur de la fourchette *Bid-Ask* et est croissant par rapport au nombre d'actifs restant à vendre. Nous comparons également nos résultats à ceux obtenus avec la stratégie "$1/n$", qui consiste à vendre à chaque date N/n actifs.

Part I

MAXIMIZATION OF UTILITY IN AN INCOMPLETE MARKET WITH DEFAULTS AND TOTAL/PARTIAL INFORMATION

Chapter 1

Exponential utility maximization and indifference pricing in an incomplete market with defaults

Joint paper with Marie-Claire Quenez.

Abstract: In this paper, we study the indifference pricing of a contingent claim via the maximization of exponential utility over a set of admissible strategies. We consider a financial market with a default time inducing a discontinuity in the price of stocks. We first consider the case of strategies valued in a *compact* set. Using a verification theorem, we show that, in the case of bounded coefficients, the value function of the exponential utility maximization problem can be characterized as *the solution of a Lipschitz BSDE* (backward stochastic differential equation). Then, we consider the case of non constrained strategies. By using dynamic programming techniques, we state that the value function is the *maximal subsolution of a BSDE*. Moreover, the value function is the *limit of a sequence of processes*, which are the value functions associated with some subsets of bounded admissible strategies. In the case of bounded coefficients, these approximating processes are the solutions of Lipschitz BSDEs, which leads to possible numerical computations. These properties can be applied to the indifference pricing problem. They can be generalized to the case of several default times or a Poisson process.

Keywords: Indifference pricing, optimal investment, exponential utility, default time, default intensity, dynamic programming principle, backward stochastic differential equation.

1.1 Introduction

In this paper, we study the indifference pricing problem in a market where the underlying traded assets are assumed to be local martingales driven by a Brownian motion and a default indicating process. We denote by $S_t = (S_t^i)_{1\leq i\leq n}$ for all $t \in [0, T]$ the price of these assets where $T < \infty$ is the fixed time horizon and n is the number of assets. The price process (S_t) is defined on a filtered space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{0\leq t\leq T}, \mathbb{P})$. Following Hodges and Neuberger [66], we define the (buying) indifference price $p(\xi)$ of a contingent claim ξ, where ξ is a $\mathcal{G}_T$-measurable random variable, as the implicit solution of the equation

$$\sup_\pi \mathbb{E}\Big[U\Big(x + \int_0^T \pi_t dS_t\Big)\Big] = \sup_\pi \mathbb{E}\Big[U\Big(x - p(\xi) + \int_0^T \pi_t dS_t + \xi\Big)\Big], \tag{1.1.1}$$

where the suprema are taken over admissible portfolio strategies π. $x \in \mathbb{R}$ is the initial endowment and U is a given utility function. In other words, the price of the contingent claim is defined as the amount of money $p(\xi)$ to withdraw to his initial wealth x that allows the investor to achieve the same supremum of the expected utility as the one he would have had with initial wealth x without buying the claim. A lot of papers study the indifference pricing problem. Among them, we quote Rouge and El Karoui [117] for a Brownian filtration, Biagini *et al.* [12] for the case of general semimartingales, Bielecki and Jeanblanc [17] for the case of a discontinuous filtration. An extensive survey of the recent literature on this topic can be found in Carmona [33].

Throughout this paper, the utility function U is assumed to be the exponential utility. By (1.1.1), the study of the indifference pricing of a given contingent claim is clearly linked to the study of the utility maximization problem.

Recall that concerning the study of the maximization of the utility of terminal wealth, there are two possible approaches:

- the first one is the *dual approach* formulated in a *static* way. This dual approach has been largely studied in the literature. Among them, in a Brownian framework, we quote Karatzas *et al.* [77] in a complete market and Karatzas *et al.* [78] in an incomplete market. In the case of general semimartingales, we quote Kramkov and Schachermayer [84], Shachermayer [121] and Delbaen *et al.* [45] for the particular case of an exponential utility function. For the case with a default in a markovian setting we refer to Lukas [94]. Using this approach, these different authors solve the utility maximization problem in the sense of finding the optimal strategy and also give a characterization of the optimal strategy via the solution of the dual problem;

- the *second approach* is the *direct* study of the primal problem(s) by using stochastic control techniques such as *dynamic programming*. Recall that these techniques had been used in finance but only in a markovian setting for along time. For example the reference paper of Merton [98] uses the well known Hamilton-Jacobi-Bellman verification theorem to solve the utility maximization problem of consumption/wealth in a complete market. The use in finance of stochastic dynamic techniques (presented in El Karoui's course [53] in a general setting) is more recent. One of the first work in finance using these techniques is that of El Karoui and Quenez [54]. Also, recall that the backward stochastic differential equations (BSDEs) have been introduced by Duffie and Epstein [49] in the case of recursive utilities and by Peng [107] for a general Lipschitz coefficient. In the paper of El Karoui *et al.* [56], several applications to finance are presented. Also, an interesting result of this paper is a *verification* theorem which allows to characterize the dynamic value function of an optimization problem as the solution of a Lipschitz BSDE. This principle stated in the Brownian case has many applications in finance. One of them can be found in Rouge and El Karoui [117] who study the exponential utility maximization problem in the incomplete Brownian case and characterize the dynamic indifference price as the solution of a quadratic BSDE (introduced by Kobylanski [83]). Concerning the exponential utility maximization problem, there is also the nice work of Hu *et al.* [67] still in the Brownian case. By using a *verification* theorem (different from the previous one), they characterize the logarithm of the dynamic value function as the solution of a quadratic BSDE.

The case of a discontinuous framework is more difficult. One reason is that there are less results on BSDEs with jumps than in the Brownian case. Concerning the study of the exponential utility maximization problem in this case, we refer to Morlais [99]. She supposes that the price process of stock is modeled by a local martingale driven by an independent Brownian motion and a Poisson point process. She mainly studies the interesting case of admissible strategies valued in a compact set (not necessarily convex). Using the same approach as in Hu *et al.* [67], she states that the logarithm of the associated value function is the unique solution of a quadratic BSDE (for which she shows an existence and a uniqueness result). In the non constrained case, she obtains formally a quadratic BSDE. She proves the existence of a solution of this BSDE by using an approximation method but she does not obtain uniqueness result. Hence, in this case, this does not allow to characterize the value function in terms of BSDEs.

In this paper, we first consider the case of strategies valued in a *compact* set. By using a verification theorem, which is a generalization of that of El Karoui *et al.* [56] to the case of jumps, we show that the value function of the exponential utility maximization problem can be characterized as the solution of a *Lipschitz BSDE*. Second, we consider the case of *non constrained* strategies. We use the dynamic programming principle to show directly that the value function is characterized as the maximal solution or the *maximal subsolution* of a BSDE. Moreover, we give *another characterization* of the value function as the *nonincreasing limit of a sequence of processes*, which are the value functions associated with some subsets of bounded admissible strategies. In the case of bounded coefficients, these approximating processes are the solutions of Lipschitz BSDEs. As a direct consequence, this suggests some possible *numerical computations* in order to approximate the value function and the indifference price. Also, we generalize these results to the case of several default times and several stocks, and to the case of a Poisson process instead of a hazard process.

The outline of this paper is organized as follows. In Section 2, we present the market model and the maximization problem in the case of only one risky asset ($n = 1$). In Section 3, we study the case of strategies valued in a *compact* set. In Section 4, we consider the *non constrained case* and state a first characterization of the value function as the maximal subsolution of a BSDE. In Section 5, we give a second characterization of the value function as the nonincreasing limit of a sequence of processes. In Section 6, we consider the classical case where the *coefficients are bounded* which simplifies the two previous characterizations of the value function. In Section 7, we study the case of unbounded *coefficients* which satisfy *some exponential integrability* conditions. Finally in Section 8, we study the *indifference price* for a contingent claim. In the last section, we generalize the previous results to the case of several assets ($n \geq 1$) and several default times, and we also extend these results to a Poisson jump model.

1.2 The market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that all processes are defined on a finite time horizon $[0, T]$. Suppose that this space is equipped with two stochastic processes: a unidimensional standard Brownian motion (W_t) and a jump process (N_t) defined by $N_t = \mathbb{1}_{\tau \leq t}$ for any $t \in [0, T]$, where τ is a random variable which modelizes a default time (see Section 1.9.1 for several default times). We assume that this default can appear at any time, that is $\mathbb{P}(\tau > t) > 0$ for any $t \in [0, T]$. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes. The filtration is supposed to be right-continuous and (W_t) is a $\mathbb{G}$-Brownian motion.

We denote by (M_t) the compensated martingale of the process (N_t) and by (Λ_t) its compensator. We assume that the compensator (Λ_t) is absolutely continuous with respect to Lebesgue's measure, so that there exists a process (λ_t) such that $\Lambda_t = \int_0^t \lambda_s ds$. Hence, the $\mathbb{G}$-martingale (M_t) satisfies

$$M_t = N_t - \int_0^t \lambda_s ds \,. \tag{1.2.1}$$

We introduce the following sets:

- $\mathcal{S}^{+,\infty}$ is the set of positive $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded càd-làg processes on $[0, T]$.
- $L^{1,+}$ is the set of positive $\mathbb{G}$-adapted càd-làg processes on $[0, T]$ such that $\mathbb{E}[Y_t] < \infty$ for any $t \in [0, T]$.

– $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{G}$-predictable processes on $[0,T]$ under $\mathbb{P}$ with

$$\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty \ \ (\text{resp. } \int_0^T |Z_t|^2 dt < \infty \ a.s.\ ).$$

– $L^2(M)$ (resp. $L^2_{loc}(M)$, $L^1_{loc}(M)$) is the set of $\mathbb{G}$-predictable processes on $[0,T]$ such that

$$\mathbb{E}\Big[\int_0^T \lambda_t |U_t|^2 dt\Big] < \infty \ \ (\text{resp. } \int_0^T \lambda_t |U_t|^2 dt < \infty, \int_0^T \lambda_t |U_t| dt < \infty \ a.s.\ ).$$

We recall the useful martingale representation theorem (see for example Jeanblanc *et al.* [73]):

Lemma 1.2.1. *Any* $(\mathbb{P},\mathbb{G})$*-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s dW_s + \int_0^t b_s dM_s, \ \forall\, t \in [0,T] \ a.s., \tag{1.2.2}$$

where $a \in L^2_{loc}(W)$ *and* $b \in L^1_{loc}(M)$*. If* $(m_t)_{0\leq t\leq T}$ *is a square integrable martingale, each term on the right-hand side of the representation (1.2.2) is square integrable.*

We now consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at any date, and one risky asset with price process S which admits a discontinuity at time τ (we give the results for n assets and p default times in Section 1.9.1). In the sequel, we consider that the price process S evolves according to the equation

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \tag{1.2.3}$$

with the classical assumptions:

Assumption 1.2.1.

(i) (μ_t), (σ_t) and (β_t) are $\mathbb{G}$-predictable processes such that $\sigma_t > 0$ and

$$\int_0^T |\sigma_t|^2 dt + \int_0^T \lambda_t |\beta_t|^2 dt < \infty \ \ a.s.,$$

(ii) the process (β_t) satisfies $\beta_\tau > -1$ (this assumption implies that the process S is positive).

We also suppose that $\mathbb{E}[\exp(-\int_0^T \alpha_t dW_t - \frac{1}{2}\int_0^T \alpha_t^2 dt)] = 1$ with $\alpha_t = (\mu_t + \lambda_t\beta_t)/\sigma_t$, which gives the existence of a martingale probability measure and hence the absence of arbitrage.

A $\mathbb{G}$-predictable process $\pi = (\pi_t)_{0\le t\le T}$ is called a trading strategy if $\int_0^T \frac{\pi_t}{S_{t^-}} dS_t$ is well defined, e.g. $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t|\pi_t\beta_t|^2 dt < \infty$ a.s. The process $(\pi_t)_{0\le t\le T}$ describes the amount of money invested in the risky asset at time t. The wealth process $(X_t^{x,\pi})$ associated with a trading strategy π and an initial capital x, under the assumption that the trading strategy is self-financing, satisfies the equation

$$\begin{cases} dX_t^{x,\pi} = \pi_t\big(\mu_t dt + \sigma_t dW_t + \beta_t dN_t\big), \\ X_0^{x,\pi} = x. \end{cases} \tag{1.2.4}$$

For a given initial time t and an initial capital x, the associated wealth process is denoted by $X_s^{t,x,\pi}$.

We assume that the investor in this financial market faces some liability, which is modeled by a random variable ξ (for example, ξ may be a contingent claim written on a default event, which itself affects the price of the underlying asset). We suppose that $\xi \in L^2(\mathcal{G}_T)$ and it is non-negative (note that all the results still hold under the assumption that ξ is only bounded from below).

Our aim is to study the classical optimization problem

$$V(x,\xi) = \sup_{\pi\in\mathcal{D}} \mathbb{E}\big[U\big(X_T^{x,\pi} + \xi\big)\big], \tag{1.2.5}$$

where $\mathcal{D}$ is a set of admissible strategies (independent of x) which will be specified in the sequel. U is an exponential utility function

$$U(x) = -\exp(-\gamma x),\ \ x \in \mathbb{R}\,,$$

where $\gamma > 0$ is a given constant, which can be seen as a coefficient of absolute risk aversion. Hence, the optimization problem (1.2.5) can be clearly written as

$$V(x,\xi) = e^{-\gamma x} V(0,\xi).$$

Hence, it is sufficient to study the case $x = 0$. To simplify notation we will denote X_t^π (resp. $X_s^{t,\pi}$) instead of $X_t^{0,\pi}$ (resp. $X_s^{t,0,\pi}$). Also, note that

$$V(0,\xi) = -\inf_{\pi\in\mathcal{D}} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^\pi + \xi\big)\big)\big]. \tag{1.2.6}$$

1.3 Strategies valued in a given compact set (in the case of bounded coefficients)

In this section, we study the case where the strategies are constrained to take their values in a compact set denoted by C (the admissible set will be denoted by $\mathcal{C}$ instead of $\mathcal{D}$).

Definition 1.3.1. The set of admissible strategies $\mathcal{C}$ is the set of predictable $\mathbb{R}$-valued processes π such that they take their values in a compact set C of $\mathbb{R}$.

We assume in this part that:

Assumption 1.3.1. The processes (μ_t), (σ_t), (β_t) and the compensator (λ_t) are uniformly bounded.

This case cannot be solved by using the dual approach because the set of admissible strategies is not necessarily convex. In this context, we address the problem of characterizing dynamically the value function associated with the exponential utility maximization problem. We give a dynamic extension of the initial problem (1.2.6) (with $\mathcal{D} = \mathcal{C}$). For any initial time $t \in [0,T]$, we define the value function $J(t,\xi)$ (also denoted by $J(t)$) by the following random variable

$$J(t,\xi) = \operatorname*{ess\,inf}_{\pi \in \mathcal{C}_t} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\big|\mathcal{G}_t\big], \tag{1.3.1}$$

where $\mathcal{C}_t$ is the set of predictable $\mathbb{R}$-valued processes π beginning at t and such that they take their values in C. Note that $V(0,\xi) = -J(0,\xi)$.

In the sequel, for ξ fixed, we want to characterize this dynamic value function $J(t)$ $(= J(t,\xi))$ as the solution of a BSDE.

For that, for each $\pi \in \mathcal{C}$, we introduce the càd-làg process (J_t^π) satisfying

$$J_t^\pi = \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\big|\mathcal{G}_t\big],\ \forall\, t \in [0,T]\,.$$

Since the coefficients are supposed to be bounded and the strategies are constrained to take their values in a compact set, it is possible to solve very simply the problem by using a *verification principle* in terms of Lipschitz BSDEs in the spirit of that of El Karoui *et al.* [56].

Note first that for any $\pi \in \mathcal{C}$, the process (J_t^π) can be easily shown to be the solution of a linear Lipschitz BSDE. More precisely, there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$, such that (J^π, Z^π, U^π) is the unique solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the linear BSDE with bounded coefficients

$$-\,dJ_t^\pi = f^\pi(t, J_t^\pi, Z_t^\pi, U_t^\pi)dt - Z_t^\pi dW_t - U_t^\pi dM_t\ ;\ J_T^\pi = \exp(-\gamma\xi), \tag{1.3.2}$$

with $f^\pi(s,y,z,u) = \frac{\gamma^2}{2}\pi_s^2\sigma_s^2 y - \gamma\pi_s(\mu_s y + \sigma_s z) - \lambda_s(1 - e^{-\gamma\pi_s\beta_s})(y+u)$.

Using the fact that $J(t) = \operatorname{ess\,inf}_{\pi\in\mathcal{C}_t} J_t^\pi$ for any $t \in [0,T]$, we state that $(J(t))$ corresponds to the solution of a BSDE, whose driver is the essential infimum over π of the drivers of BSDEs (1.3.2). More precisely,

Proposition 1.3.1. *The following properties hold:*

– *Let (Y, Z, U) be the solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the following BSDE*

$$\begin{cases} -dY_t = \operatorname*{ess\,inf}_{\pi \in \mathcal{C}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 Y_t - \gamma \pi_t (\mu_t Y_t + \sigma_t Z_t) - \lambda_t (1 - e^{-\gamma \pi_t \beta_t})(Y_t + U_t) \Big\} dt \\ \qquad\quad - Z_t dW_t - U_t dM_t \,, \\ Y_T = \exp(-\gamma \xi) \,. \end{cases} \tag{1.3.3}$$

Then, for any $t \in [0, T]$, $J(t) = Y_t$ a.s.

– *There exists a unique optimal strategy $\hat{\pi} \in \mathcal{C}$ for $J(0) = \inf_{\pi \in \mathcal{C}} \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$, and this strategy is characterized by the fact that it attains the essential infimum in (1.3.3) $dt \otimes d\mathbb{P} - a.e.$*

Proof. Let us introduce the driver f which satisfies $ds \otimes d\mathbb{P} - a.e.$

$$f(s, y, z, u) = \operatorname*{ess\,inf}_{\pi \in \mathcal{C}} f^\pi(s, y, z, u).$$

Since the driver f is written as an infimum of linear drivers $(f^\pi)_{\pi \in \mathcal{C}}$ w.r.t (y, z, u) with uniformly bounded coefficients (by assumption), f is clearly Lipschitz (see Lemma 1.10.1 in Appendix 1.10.2). Hence, by Tang and Li's results [126], BSDE (1.3.3) with Lipschitz driver f

$$-dY_t = f(t, Y_t, Z_t, U_t)dt - Z_t dW_t - U_t dM_t \ ; \ Y_T = \exp(-\gamma \xi)$$

admits a unique solution denoted by (Y, Z, U).

Since, we have

$$f^\pi(t, y, z, u) - f^\pi(t, y, z, u') = \lambda_t (u - u') \gamma^t, \tag{1.3.4}$$

with $\gamma^t = e^{-\gamma \pi_t \beta_t} - 1$, and since there exist some constants $-1 < C_1 \leq 0$ and $0 \leq C_2$ such that $C_1 \leq \gamma^t \leq C_2$, the comparison theorem in case of jumps (see for example Theorem 2.5 in Royer [118]) can be applied and implies that $Y_t \leq J_t^\pi$, $\forall\, t \in [0, T]$ a.s. As this inequality is satisfied for any $\pi \in \mathcal{C}$, it is obvious that $Y_t \leq \operatorname{ess\,inf}_{\pi \in \mathcal{C}} J_t^\pi$ a.s. Also, by applying a measurable selection theorem, one can easily show that there exists $\hat{\pi} \in \mathcal{C}$ such that $dt \otimes d\mathbb{P}$-a.s.

$$\begin{aligned} &\operatorname*{ess\,inf}_{\pi \in \mathcal{C}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 Y_t - \gamma \pi_t (\mu_t Y_t + \sigma_t Z_t) - \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(Y_t + U_t) \Big\} \\ &\qquad\qquad = \frac{\gamma^2}{2} \hat{\pi}_t^2 \sigma_t^2 Y_t - \gamma \hat{\pi}_t (\mu_t Y_t + \sigma_t Z_t) - \lambda_t \big(1 - e^{-\gamma \hat{\pi}_t \beta_t}\big)(Y_t + U_t). \end{aligned}$$

Thus, (Y, Z, U) is a solution of BSDE (1.3.2) associated with $\hat{\pi}$. Therefore, by uniqueness of the solution of BSDE (1.3.2), we have $Y_t = J_t^{\hat{\pi}}$, $\forall\, t \in [0, T]$ a.s. Hence, $Y_t = \operatorname{ess\,inf}_{\pi \in \mathcal{C}_t} J_t^{\pi} = J_t^{\hat{\pi}}$, $\forall\, t \in [0, T]$ a.s., and $\hat{\pi}$ is an optimal strategy. It is obvious that the optimal strategy is unique because the function $x \mapsto \exp(-\gamma x)$ is strictly convex. □

Remark 1.3.1. The proof is short and simple thanks to the *verification principle* of BSDEs and optimization. Note that this *verification principle* is similar to the one stated in the Brownian case by El Karoui *et al.* [56] but needs some particular conditions on the coefficients (see (1.3.4)) due to the presence of defaults.

Remark 1.3.2. Note that this problem has already been studied by Morlais [99]. By using a verification theorem similar to that of Hu *et al.* [67], she states that the logarithm of the value function is the unique solution of a quadratic BSDE. In order to obtain this characterization, she proves the existence and the uniqueness of a solution for this quadratic BSDE with jumps by using a quite sophisticated approximation method in the spirit of Kobylanski [83].

Note that by making a change of variables, the above proposition (Proposition 1.3.1) corresponds to Morlais's result [99]. Indeed, put

$$\begin{cases} y_t = \dfrac{1}{\gamma} \log(Y_t), \\ z_t = \dfrac{1}{\gamma} \dfrac{Z_t}{Y_t}, \\ u_t = \dfrac{1}{\gamma} \log\left(1 + \dfrac{U_t}{Y_{t^-}}\right), \end{cases}$$

it is clear that the process (y, z, u) is the solution of the following quadratic BSDE

$$-\,dy_t = g(t, z_t, u_t)dt - z_t dW_t - u_t dM_t \ ; \ y_T = -\xi \ ,$$

with

$$g(s, z, u) = \inf_{\pi \in \mathcal{C}} \left(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s + \lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u - \pi_s\beta_s|_\gamma\right) - (\mu_s + \lambda_s\beta_s)z - \frac{|\mu_s + \lambda_s\beta_s|^2}{2\gamma},$$

which corresponds exactly to Morlais's result [99] with $|u - \pi\beta_t|_\gamma = \lambda_t \frac{\exp(\gamma(u - \pi\beta_t)) - 1 - \gamma(u - \pi\beta_t)}{\gamma}$.

This characterization of the value function as the solution of a Lipschitz BSDE leads to possible numerical computations of the value function (see for example Bouchard and Elie [22]) and of the indifference price defined via this utility maximization problem (see Section 1.8).

Moreover, this property will be used to state that in the non constrained case, the value function can be approximated by a sequence of Lipschitz BSDEs (see Theorem 1.7.2).

1.4 The non constrained case: characterization of the value function by a BSDE

In this section, the coefficients are no longer supposed to be bounded. We now study the value function in the case where the admissible strategies are no longer required to satisfy any constraints (as in the previous section). Since the utility function is the exponential utility function, the set of admissible strategies is not standard in the literature. The next subsection studies the choice of a suitable set of admissible strategies which will allow to dynamize the problem and to characterize the associated value function (and even the dynamic value function).

1.4.1 The set of admissible strategies

Recall that in the case of the power or logarithmic utility functions defined (or restricted) on $\mathbb{R}^+$, the admissible strategies are the ones that make the associated wealth positive. Since we consider the exponential utility function $U(x) = -\exp(-\gamma x)$ which is finitely valued for all $x \in \mathbb{R}$, the wealth process is no longer required to be positive. However, it is natural to consider strategies such that the associated wealth process is uniformly bounded by below (see for example Schachermayer [121]) or even such that any increment of the wealth is bounded by below. More precisely,

Definition 1.4.1. The set of admissible trading strategies $\mathcal{A}$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{0\leq t\leq T}$, which satisfy $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t|\pi_t\beta_t|^2 dt < \infty$ a.s., and such that for any π fixed and any $s \in [0,T]$, there exists a real constant $K_{s,\pi}$ such that $X_t^\pi - X_s^\pi \geq -K_{s,\pi}$, $s \leq t \leq T$ a.s.

Recall that in their paper, Delbaen *et al.* [45] also consider the two following sets of strategies:

- the set Θ_3 of strategies such that the wealth process is bounded,
- the set Θ_2 defined by

$$\Theta_2 := \Big\{\pi\,,\, \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi + \xi)\big)\big] < +\infty \text{ and } X^\pi \text{ is a } \mathbb{Q}-\text{martingale for all } \mathbb{Q} \in \mathbb{P}_f\Big\},$$

  where $\mathbb{P}_f$ is the set of absolutely continuous local martingale measures $\mathbb{Q}$ such that its entropy $H(\mathbb{P}|\mathbb{Q})$ is finite.

Note that $\Theta_3 \subset \mathcal{A}$. Of course, there is no existence result neither for the space Θ_3 nor for $\mathcal{A}$ whereas there is one on the set Θ_2 stated by Delbaen *et al.* [45]. More precisely, by using the dual approach, under the assumption that the price process is locally bounded, these authors show the existence of an optimal strategy on the set Θ_2. Also, they stress on the following important point: under the assumption that the price process is locally bounded (which is satisfied if for example β is bounded), the value function associated with Θ_2 coincides with that associated with Θ_3. From this, we easily derive that these value functions also coincide with that associated with $\mathcal{A}$. More precisely,

Lemma 1.4.1. *Suppose that the process (β_t) is bounded. The value function $V(0, \xi)$ associated with $\mathcal{A}$ defined by*

$$V(0, \xi) = -\inf_{\pi \in \mathcal{A}} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^\pi + \xi\big)\big)\big] \tag{1.4.1}$$

is equal to the one associated with Θ_2 (and also the one associated with Θ_3).

Proof. By the result of Delbaen *et al.* [45], the value function associated with Θ_2 coincides with that associated with Θ_3 denoted by $V^3(0, \xi)$. Now, since $\Theta_3 \subset \mathcal{A}$, we have $V(0, \xi) \geq V^3(0, \xi)$. By a localization argument (such as in the proof of Lemma 1.4.3), one can easily show the equality, which gives the desired result. □

Our aim is mainly to characterize and even to compute or approximate the value function $V(0, \xi)$.

Our approach consists in giving a dynamic extension of the optimization problem and in using stochastic calculus techniques in order to characterize the dynamic value function. In the compact case (with the set $\mathcal{C}$), the dynamic extension was easy (see Section 1.3). At any initial time t, the corresponding set $\mathcal{C}_t$ of admissible strategies was simply given by the set of the restrictions to $[t, T]$ of the strategies of $\mathcal{C}$. In the case of $\mathcal{A}$ or Θ_3, it is also very simple (see below for $\mathcal{A}$). However, in the case of the set Θ_2, things are not so clear. Actually, this is partly linked to the fact that, contrary to the set Θ_2, the set $\mathcal{A}$ is closed by *binding*. More precisely, we clearly have:

Lemma 1.4.2. *The set $\mathcal{A}$ is* closed by *binding that is: if π^1, π^2 are two strategies of $\mathcal{A}$ and if $s \in [0, T]$, then the strategy π^3 defined by*

$$\pi_t^3 = \begin{cases} \pi_t^1 & \text{if } t \leq s, \\ \pi_t^2 & \text{if } t > s, \end{cases}$$

belongs to $\mathcal{A}$.

Also, the set Θ_2 is clearly *not closed by binding* because of the integrability condition $\mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))] < +\infty$. One could naturally think of considering the space $\Theta_2' := \{\pi\,,\ \ X^\pi \text{ is a } \mathbb{Q} - \text{martingale for all } \mathbb{Q} \in \mathbb{P}_f\}$ (instead of Θ_2) but this set is not really appropriate: in particular it does not allow to obtain the dynamic programming principle since the Lebesgue theorem cannot be applied (see Remark 1.4.2).

However, there are some other possible sets which are closed by binding as for example

- the set Θ_3 of strategies such that the wealth process is bounded,
- the set $\mathcal{A}'$ defined as the set of $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{0\leq t\leq T}$ with $\int_0^T |\pi_t\sigma_t|^2dt + \int_0^T \lambda_t|\pi_t\beta_t|^2dt < \infty$ a.s., and such that for any $t \in [0,T]$ and for any $p > 1$, the following integrability condition

$$\mathbb{E}\Big[\sup_{s\in[t,T]} \exp\Big(-\gamma p X_s^{t,\pi}\Big)\Big] < \infty \tag{1.4.2}$$

  holds.

Note that $\Theta_3 \subset \mathcal{A} \subset \mathcal{A}'$.

Remark 1.4.1. Note that in general, there is no existence result for the set $\mathcal{A}'$.

For the proof of the closedness by binding of the set $\mathcal{A}'$ one is referred to Appendix 1.10.3. Note that in this proof, we see that the integrability condition $\mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))] < +\infty$ is not sufficient to derive this closedness property by binding. It is the assumption of p-integrability (1.4.2) for $p > 1$ (and not only the integrability) which allows to derive the desired property. Note that this type of p-exponential integrability condition appears in some papers related to quadratic BSDEs.

Let us now give a *dynamic extension* of the initial problem associated with $\mathcal{A}$ given by (1.4.1). For any initial time $t \in [0,T]$, we define the value function $J(t,\xi)$ by the following random variable

$$J(t,\xi) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\big|\mathcal{G}_t\big], \tag{1.4.3}$$

where the set $\mathcal{A}_t$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_s)_{t\leq s\leq T}$, which satisfy $\int_t^T |\pi_s\sigma_s|^2ds + \int_t^T \lambda_s|\pi_s\beta_s|^2ds < \infty$ a.s., and such that for any π fixed and any $s \in [t,T]$ there exists a constant $K_{s,\pi}$ such that $X_u^{s,\pi} \geq -K_{s,\pi}$, $s \leq u \leq T$ a.s.
Note that $J(0,\xi) = -V(0,\xi)$. Also, for any $t \in [0,T]$, $J(t,\xi)$ is also equal a.s. to the essinf in (1.4.3) but taken over $\mathcal{A}$ instead of $\mathcal{A}_t$. This clearly follows from the fact that the set $\mathcal{A}_t$ is equal to the set of the restrictions to $[t,T]$ of the strategies of $\mathcal{A}$.
For the sake of brevity, we shall denote $J(t)$ instead of $J(t,\xi)$. Note that the random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent. The process $(J(t))$ will be called the *dynamic value function.* This process is adapted but not necessarily càd-làg

and not even progressive.
Similarly, a dynamic extension of the value function associated with $\mathcal{A}'$ (or also Θ_3) can be easily given. Under the assumption that the price process is locally bounded (which is satisfied if for example β is bounded), the corresponding value functions can be easily shown to coincide a.s. More precisely,

Lemma 1.4.3. *Suppose that the coefficient (β_t) is bounded. The dynamic value function $(J(t))$ associated with $\mathcal{A}$ coincides a.s. with the one associated with $\mathcal{A}'$ (or also Θ_3).*

Proof. We give here the proof for $\mathcal{A}'$ (it is the same for Θ_3). Fix $t \in [0,T]$. Put $J^{'}(t) := \operatorname{ess\,inf}_{\pi \in \mathcal{A}_t^{'}} \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathcal{G}_t]$, where $\mathcal{A}_t^{'}$ is the set defined similarly as $\mathcal{A}^{'}$ but for initial time t. Note that $\mathcal{A}_t^{'}$ can be seen as the set of the restrictions to $[t,T]$ of the strategies of $\mathcal{A}^{'}$. Since $\mathcal{A}_t \subset \mathcal{A}_t^{'}$, we get $J^{'}(t) \leq J(t)$. To prove the other inequality, we state that for any $\pi \in \mathcal{A}_t^{'}$, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{A}_t$ such that $\pi^n \to \pi,\ dt \otimes d\mathbb{P}$ a.s. Let us define π^n by

$$\pi_s^n = \pi_s \mathbb{1}_{s \leq \tau_n},\ \forall\ s \in [t,T],$$

where τ_n is the stopping time defined by $\tau_n = \inf\{s \geq t, |X_s^{t,\pi}| \geq n\}$.
It is clear that for each $n \in \mathbb{N}$, $\pi^n \in \mathcal{A}_t$. Thus, $\exp(-\gamma X_T^{t,\pi^n}) = \exp(-\gamma X_{T\wedge\tau_n}^{t,\pi}) \xrightarrow{a.s.} \exp(-\gamma X_T^{t,\pi})$ as $n \to +\infty$. By definition of $\mathcal{A}_t^{'}$, $\mathbb{E}[\sup_{s\in[t,T]} \exp(-\gamma X_s^{t,\pi})] < \infty$. Hence, by the Lebesgue theorem, $\mathbb{E}[\exp(-\gamma(X_T^{t,\pi^n} + \xi))|\mathcal{G}_t] \to \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathcal{G}_t]$ a.s. as $n \to +\infty$. Therefore, we have $J(t) \leq J^{'}(t)$ a.s., which ends the proof. □

Hence, concerning the dynamic study of the value function, if (β_t) is supposed to be bounded, it is equivalent to choose $\mathcal{A}$, $\mathcal{A}'$ or Θ_3 as set of admissible strategies. We have chosen the set $\mathcal{A}$ because it appears as a natural set of admissible strategies from a financial point of view.

After this dynamic extension of the value function, we will use stochastic calculus techniques in order to characterize the value function via a BSDE. However, it is no longer possible to use a verification theorem like the one in Section 1.3 because the associated BSDE is no longer Lipschitz and there is no existence result for it. One could think to use a verification theorem like that of Hu *et al.* [67]. But because of the presence of jumps, it is no longer possible since again there is no existence and uniqueness results for the associated BSDE as noted by Morlais [99]. In her paper, Morlais proves the existence of a solution of this BSDE by using an approximation method but she does not obtain uniqueness result, even in the case of bounded coefficients. Hence, this does not a priori lead to a characterization of the value function via a BSDE.

Therefore, as it seems not possible to derive a *sufficient condition* so that a given process corresponds to the dynamic value function, we will now directly study some properties of the dynamic value function $(J(t))$ (in other words some *necessary conditions* satisfied by $(J(t))$). Then, by using dynamic programming techniques of stochastic control, we will derive a characterization of the value function via a BSDE. This is the object of the next section.

1.4.2 Characterization of the dynamic value function as the maximal subsolution of a BSDE

The dynamic programming principle holds for the set $\mathcal{A}$:

Proposition 1.4.1. *The process* $(\exp(-\gamma X_t^\pi)J(t))_{0\leq t\leq T}$ *is a submartingale for any* $\pi \in \mathcal{A}$.

To prove this proposition, we use the random variables $(J_t^\pi)_{\pi\in\mathcal{A}_t}$ which are defined for any $\pi \in \mathcal{A}_t$ by

$$J_t^\pi = \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi}+\xi\big)\big)\big|\mathcal{G}_t\big].$$

As usual, in order to prove the dynamic programming principle, we first state the following lemma:

Lemma 1.4.4. *The set* $\{J_t^\pi,\ \pi \in \mathcal{A}_t\}$ *is* stable by pairwise minimization *for any* $t \in [0,T]$. *That is, for every* $\pi^1,\ \pi^2 \in \mathcal{A}_t$, *there exists* $\pi \in \mathcal{A}_t$ *such that* $J_t^\pi = J_t^{\pi^1} \wedge J_t^{\pi^2}$.
Also, there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{A}_t$ *for any* $t \in [0,T]$, *such that*

$$J(t) = \lim_{n\to\infty} \downarrow J_t^{\pi^n} \ a.s.$$

Proof. Fix $t \in [0,T]$. Let us introduce the set $E = \{J_t^{\pi^1} \leq J_t^{\pi^2}\}$ which belongs to $\mathcal{G}_t$. Let us define π for any $s \in [t,T]$ by $\pi_s = \pi_s^1 \mathbb{1}_E + \pi_s^2 \mathbb{1}_{E^c}$. It is obvious that $\pi \in \mathcal{A}_t$, since the sum of two random variables bounded by below is bounded by below. By construction of π, it is clear that $J_t^\pi = J_t^{\pi^1} \wedge J_t^{\pi^2}$.
The second part of lemma follows by classical results on the essential infimum (see Appendix 1.10.1). □

Let us now give the proof of Proposition 1.4.1.

Proof. Let us show that for $t \geq s$,

$$\mathbb{E}\big[\exp\big(-\gamma\big(X_t^\pi - X_s^\pi\big)\big)J(t)\big|\mathcal{G}_s\big] \geq J(s) \ a.s.$$

Note that $X_t^\pi - X_s^\pi = X_t^{s,\pi}$. By Lemma 1.4.4, there exists a sequence $(\pi_n)_{n\in\mathbb{N}} \in \mathcal{A}_t$ such that $J(t) = \lim_{n\to\infty} \downarrow J_t^{\pi^n}$ a.s.

Without loss of generality, we can suppose that $\pi^0 = 0$. For each $n \in \mathbb{N}$, we have $J_t^{\pi^n} \leq J_t^{\pi^0} \leq 1$ a.s. Moreover, the integrability property $\mathbb{E}[\exp(-\gamma X_t^{s,\pi})] < \infty$ holds because $\pi \in \mathcal{A}$. This with the Lebesgue theorem give

$$\mathbb{E}\big[\lim_{n\to\infty} \exp\big(-\gamma X_t^{s,\pi}\big) J_t^{\pi^n} \big| \mathcal{G}_s\big] = \lim_{n\to\infty} \mathbb{E}\big[\exp\big(-\gamma X_t^{s,\pi}\big) J_t^{\pi^n} \big| \mathcal{G}_s\big]. \tag{1.4.4}$$

Recall that $X_t^{s,\pi} = \int_s^t \frac{\pi_u}{S_{u^-}} dS_u$. Now, we have a.s.

$$\exp\Big(-\gamma \int_s^t \frac{\pi_u}{S_{u^-}} dS_u\Big) J_t^{\pi^n} = \mathbb{E}\Big[\exp\Big(-\gamma\Big(\int_s^T \frac{\tilde{\pi}_u^n}{S_{u^-}} dS_u + \xi\Big)\Big) \Big| \mathcal{G}_t\Big], \tag{1.4.5}$$

where the strategy $\tilde{\pi}^n$ is defined by

$$\tilde{\pi}_u^n = \begin{cases} \pi_u & \text{if } 0 \leq u \leq t, \\ \pi_u^n & \text{if } t < u \leq T. \end{cases}$$

Note that by the closedness property by binding (see Lemma 1.4.2), $\tilde{\pi}^n \in \mathcal{A}$ for each $n \in \mathbb{N}$. By (1.4.4) and (1.4.5), we have a.s.

$$\begin{aligned} \mathbb{E}\Big[\exp\Big(-\gamma \int_s^t \frac{\pi_u}{S_{u^-}} dS_u\Big) J(t) \Big| \mathcal{G}_s\Big] &= \lim_{n\to\infty} \mathbb{E}\Big[\exp\Big(-\gamma\Big(\int_s^T \frac{\tilde{\pi}_u^n}{S_{u^-}} dS_u + \xi\Big)\Big) \Big| \mathcal{G}_s\Big] \\ &= \lim_{n\to\infty} J_s^{\tilde{\pi}^n} \quad \geq \quad J(s), \end{aligned}$$

because by definition of $J(s)$, we have $J_s^{\tilde{\pi}^n} \geq J(s)$ a.s., for each $n \in \mathbb{N}$. Hence, the process $(\exp(-\gamma X_t^\pi) J(t))$ is a submartingale for any $\pi \in \mathcal{A}$. □

Remark 1.4.2. Note that the integrability property $\mathbb{E}[\exp(-\gamma X_t^{s,\pi})] < \infty$ is essential in the proof of this property. Indeed, if it is not satisfied, equality (1.4.4) does not hold since the Lebesgue theorem cannot be applied. One could argue that the monotone convergence theorem could be used but since the limit is decreasing, it cannot be applied without an integrability condition. Moreover, Fatou's lemma is not relevant since it gives an inequality but not in the suitable sense. Actually, the importance of the integrability condition is due to the fact that we study an *essential infimum* of positive random variables. In the case of an *essential supremum* of positive random variables, the dynamic programming principle holds without any integrability condition (see for example the case of the power utility function in Lim and Quenez [93]).

Also, the value function can easily be characterized as follows:

Proposition 1.4.2. *The process* $(J(t))$ *is the largest* $\mathbb{G}$*-adapted process such that* $(e^{-\gamma X_t^\pi} J(t))$ *is a submartingale for any admissible strategy* $\pi \in \mathcal{A}$ *with* $J(T) = \exp(-\gamma\xi)$*. More precisely, if* $(\hat{J}_t)$ *is a* $\mathbb{G}$*-adapted process such that* $(\exp(-\gamma X_t^\pi)\hat{J}_t)$ *is a submartingale for any* $\pi \in \mathcal{A}$ *with* $\hat{J}_T = \exp(-\gamma\xi)$*, then we have* $J(t) \geq \hat{J}_t$ *a.s., for any* $t \in [0, T]$*.*

Proof. Fix $t \in [0, T]$. For any $\pi \in \mathcal{A}$, $\mathbb{E}[\exp(-\gamma X_T^\pi)\hat{J}_T|\mathcal{G}_t] \geq \exp(-\gamma X_t^\pi)\hat{J}_t$ a.s. This implies

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\big|\mathcal{G}_t\big] \geq \hat{J}_t \quad a.s.,$$

which gives clearly that $J(t) \geq \hat{J}_t$ a.s. □

With this property, it is possible to show that there exists a càd-làg version of the value function $(J(t))$. More precisely, we have:

Proposition 1.4.3. *There exists a* $\mathbb{G}$*-adapted càd-làg process* (J_t) *such that for any* $t \in [0, T]$*,*

$$J_t = J(t) \quad a.s.$$

A direct proof is given in Appendix 1.10.4.

Remark 1.4.3. Note that Proposition 1.4.2 can be written under the form: (J_t) is the largest $\mathbb{G}$-adapted càd-làg process such that the process $(\exp(-\gamma X_t^\pi)J_t)$ is a submartingale for any $\pi \in \mathcal{A}$ with $J_T = \exp(-\gamma\xi)$.

We now prove that the process (J_t) is bounded. More precisely, we have:

Lemma 1.4.5. *The process* (J_t) *verifies*

$$0 \leq J_t \leq 1, \ \forall\, t \in [0, T] \ a.s.$$

Proof. Fix $t \in [0, T]$. The first inequality is easy to prove, because it is obvious that

$$0 \leq \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\big|\mathcal{G}_t\big] \ a.s.,$$

for any $\pi \in \mathcal{A}_t$, which implies $0 \leq J_t$.

The second inequality is due to the fact that the strategy defined by $\pi_s = 0$ for any $s \in [t, T]$ is admissible, which implies $J_t \leq \mathbb{E}[\exp(-\gamma\xi)|\mathcal{G}_t]$ a.s. As the contingent claim ξ is supposed to be non negative, we have $J_t \leq 1$ a.s. □

Remark 1.4.4. Note that if ξ is only bounded by below by a real constant $-K$, then (J_t) is still upper bounded but by $\exp(\gamma K)$ instead of 1.

In our setting, it is not possible to use the verification theorem of Section 1.3 or even the verification theorem of Hu *et al.* [67] in the Brownian case. Using the previous characterization of the value function (see Proposition 1.4.2), we will show directly that the value function (J_t) is characterized by a BSDE. Since we work in terms of *necessary conditions* satisfied by the value function, the study is more technical than in the cases where a verification theorem can be applied.

Since (J_t) is a càd-làg submartingale and is bounded (see Lemma 1.4.5), and hence of class D, it admits a unique Doob-Meyer decomposition (see Dellacherie and Meyer [46], Chapter 7)

$$dJ_t = dm_t + dA_t\,,$$

where (m_t) is a square integrable martingale and (A_t) is an increasing $\mathbb{G}$-predictable process with $A_0 = 0$. From the martingale representation theorem (see Proposition 1.2.1), the previous Doob-Meyer decomposition can be written under the form

$$dJ_t = Z_t dW_t + U_t dM_t + dA_t\,, \tag{1.4.6}$$

with $Z \in L^2(W)$ and $U \in L^2(M)$.
Using the dynamic programming principle, it is possible to precise the process (A_t) of (1.4.6). This allows to show that the value function (J_t) is a subsolution of a BSDE. For that, we define the set $\mathcal{A}^2$ of the increasing adapted càd-làg processes K such that $K_0 = 0$ and $\mathbb{E}|K_T|^2 < \infty$. More precisely,

Proposition 1.4.4. *There exists a process $K \in \mathcal{A}^2$ such that the process $(J, Z, U, K) \in \mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$ is a subsolution of the following BSDE*

$$\begin{cases} -\,dJ_t = \displaystyle\operatorname*{ess\,inf}_{\pi\in\mathcal{A}} \Big\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t - \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) - \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(J_t + U_t)\Big\}dt \\ \qquad\quad -\, dK_t - Z_t dW_t - U_t dM_t\,, \\ J_T = \exp(-\gamma\xi)\,. \end{cases} \tag{1.4.7}$$

Proof. The proof of this proposition is based on the dynamic programming principle: the process $(\exp(-\gamma X_t^\pi)J_t)$ is a submartingale for any $\pi \in \mathcal{A}$ (see Proposition 1.4.2). First, we write the derivative of $\exp(-\gamma X_t^\pi)J_t$ under the following form

$$d(e^{-\gamma X_t^\pi} J_t) = dA_t^\pi + dm_t^\pi,$$

with $A_0^\pi = 0$ and

$$\begin{cases} dA_t^\pi &= e^{-\gamma X_t^\pi}\Big[dA_t + \Big\{\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t) - \gamma\pi_t(\sigma_t Z_t + \mu_t J_t)\Big\}dt\Big], \\ dm_t^\pi &= e^{-\gamma X_{t^-}^\pi}\big[(Z_t - \gamma\pi_t\sigma_t J_t)dW_t + (U_t + (e^{-\gamma\pi_t\beta_t}-1)(U_t+J_{t^-}))dM_t\big]. \end{cases}$$

Since for any $\pi \in \mathcal{A}$ the process $(\exp(-\gamma X_t^\pi)J_t)$ is a submartingale, we have

$$dA_t \geq \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t) + \gamma\pi_t(\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t\Big\}dt. \tag{1.4.8}$$

We define the process (K_t) by $K_0 = 0$ and

$$dK_t = dA_t - \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t) + \gamma\pi_t(\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t\Big\}dt.$$

It is clear that the process (K_t) is nondecreasing from (1.4.8). Since the strategy defined by $\pi_t = 0$ for any $t \in [0, T]$ is admissible, we have

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t) + \gamma\pi_t(\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t\Big\} \geq 0.$$

Hence, $0 \leq K_t \leq A_t$ a.s. As $\mathbb{E}|A_T|^2 < \infty$, we have $K \in \mathcal{A}^2$. Thus, the Doob-Meyer decomposition (1.4.6) of (J_t) can be written as follows

$$\begin{aligned} dJ_t = &\operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t) + \gamma\pi_t(\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t\Big\}dt \\ &+ dK_t + Z_t dW_t + U_t dM_t, \end{aligned}$$

with $Z \in L^2(W)$, $U \in L^2(M)$ and $K \in \mathcal{A}^2$. This ends the proof. □

The fact that (J, Z, U, K) is a subsolution of BSDE (1.4.7) does not allow to characterize the value function, since the subsolution of BSDE (1.4.7) is not unique. However, we have the following characterization of the value function:

Theorem 1.4.1. *(Characterization of the value function)*
(J, Z, U, K) is the maximal subsolution *in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$ of BSDE (1.4.7). That is for any subsolution $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ of the BSDE in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$, we have $\bar{J}_t \leq J_t$, $\forall\, t \in [0, T]$ a.s.*

Remark 1.4.5. *If ξ and the coefficients are supposed to be bounded, we will see, in Section 1.6, that (J, Z, U) is the maximal solution of BSDE (1.4.7), that is with $K_t = 0$ for any $t \in [0, T]$ (see Theorem 1.6.2).*

Proof. Let $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ be a subsolution of (1.4.7) in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$. Let us prove that the process $(\exp(-\gamma X_t^\pi)\bar{J}_t)$ is a submartingale for any $\pi \in \mathcal{A}$.
From the product rule, we can write the derivative of this process under the form

$$d\big(e^{-\gamma X_t^\pi}\bar{J}_t\big) = d\bar{M}_t^\pi + d\bar{A}_t^\pi + e^{-\gamma X_t^\pi} d\bar{K}_t,$$

with $\bar{A}_0^\pi = 0$ and

$$\begin{cases} d\bar{A}_t = -\underset{\pi\in\mathcal{A}}{\operatorname{ess\,inf}} \Big\{\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}dt, \\ d\bar{A}_t^\pi = e^{-\gamma X_t^\pi}\Big\{\Big[\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big]dt + d\bar{A}_t\Big\}, \\ d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi}\big[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t)dW_t + \big(\bar{U}_t + (e^{-\gamma\pi_t\beta_t}-1)\big(\bar{U}_t + \bar{J}_{t^-}\big)\big)dM_t\big]. \end{cases}$$

Since the strategy π is admissible, there exists a constant C_π such that $\exp(-\gamma X_t^\pi) \le C_\pi$ for any $t \in [0,T]$. With this, one can easily derive that $\mathbb{E}[\sup_{t\in[0,T]}\exp(-\gamma X_t^\pi)\bar{J}_t] < +\infty$ and that $\mathbb{E}[\int_0^T \exp(-\gamma X_t^\pi)d\bar{K}_t] < +\infty$. It follows that the local martingale $(\bar{M}_t^\pi)$ is a martingale and that the process $(\exp(-\gamma X_t^\pi)\bar{J}_t)$ is a submartingale.
Now recall that (J_t) is the largest process such that $(\exp(-\gamma X_t^\pi)J_t)$ is a submartingale for any $\pi \in \mathcal{A}$ with $J_T = \exp(-\gamma\xi)$ (see Proposition 1.4.2). Therefore, we get

$$\bar{J}_t \le J_t, \ \forall\, t \in [0,T] \ a.s.$$

□

Remark 1.4.6. Note that the integrability property $\mathbb{E}[\sup_{t\in[0,T]}\exp(-\gamma X_t^\pi)]$ is essential in this proof.

1.5 The non constrained case: approximation of the value function

In this section, we do not make any assumptions on the coefficients of the model.
In the sequel, the value function is shown to be characterized as the *limit of a nonincreasing sequence* of processes $((J_t^k))_{k\in\mathbb{N}}$ as k tends to $+\infty$, where for each $k \in \mathbb{N}$, (J_t^k) corresponds to the value function over the set of admissible strategies bounded by k.

Note that in the classical case of *bounded coefficients*, we will see in the next section that for each $k \in \mathbb{N}$, (J_t^k) can be characterized as the solution of a *Lipschitz BSDE*.

For each $k \in \mathbb{N}$, we denote by $\mathcal{A}_t^k$ the set of strategies of $\mathcal{A}_t$ uniformly bounded by k, and we consider the associated value function $J^k(t)$ defined by

$$J^k(t) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t^k} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi} + \xi\big)\big)\big|\mathcal{G}_t\big]. \tag{1.5.1}$$

By similar argument as for (J_t), there exists a càd-làg version of $(J^k(t))$ denoted by (J_t^k). As previously, the dynamic programming principle holds:

Proposition 1.5.1. *The process* $(\exp(-\gamma X_t^\pi)J_t^k)$ *is a submartingale for any* $\pi \in \mathcal{A}^k$.

We now show that the sequence of value functions $((J_t^k))_{k\in\mathbb{N}}$ converges to the value function (J_t). More precisely, we have:

Theorem 1.5.1. *(Approximation of the value function)*
For any $t \in [0, T]$, *we have*

$$J_t = \lim_{k\to\infty} \downarrow J_t^k \quad a.s.$$

Proof. Fix $t \in [0, T]$. It is obvious with the definitions of sets $\mathcal{A}_t$ and $\mathcal{A}_t^k$ that $\mathcal{A}_t^k \subset \mathcal{A}_t$ for each $k \in \mathbb{N}$, and hence

$$J_t \leq J_t^k \ a.s.$$

Moreover, since $\mathcal{A}_t^k \subset \mathcal{A}_t^{k+1}$ for each $k \in \mathbb{N}$, it follows that the sequence of positive random variables $(J_t^k)_{k\in\mathbb{N}}$ is nonincreasing. Let us define the random variable

$$\bar{J}(t) = \lim_{k\to\infty} \downarrow J_t^k \ a.s.$$

It is obvious from the previous inequality that $J_t \leq \bar{J}(t)$ a.s., and this holds for any $t \in [0, T]$. It remains to prove that $J_t \geq \bar{J}(t)$ a.s. for any $t \in [0, T]$. This will be done by the following steps.

Step 1: Let us now prove that the process $(\bar{J}(t))$ is a submartingale. Fix $0 \leq s < t \leq T$. From Proposition 1.5.1, (J_t^k) is a submartingale, which gives for each $k \in \mathbb{N}$

$$\mathbb{E}\big[J_t^k\big|\mathcal{G}_s\big] \geq J_s^k \geq \bar{J}(s) \ a.s.$$

The dominated convergence theorem (which can be applied since $0 \leq J_t^k \leq 1$ for each $k \in \mathbb{N}$) gives

$$\mathbb{E}\big[\bar{J}(t)\big|\mathcal{G}_s\big] = \lim_{k\to\infty} \mathbb{E}\big[J_t^k\big|\mathcal{G}_s\big] \geq \bar{J}(s) \ a.s.,$$

which gives step 1.

Step 2: Let us show that the process $(\exp(-\gamma X_t^\pi)\bar{J}(t))$ is a submartingale for any bounded strategy $\pi \in \mathcal{A}$.
Let π be a bounded admissible strategy. Then, there exists $n \in \mathbb{N}$ such that π is uniformly bounded by n. For each $k \geq n$, since $\pi \in \mathcal{A}^k$, $(\exp(-\gamma X_t^\pi)J_t^k)$ is a submartingale from Proposition 1.5.1. Then, by the dominated convergence theorem, the process $(\exp(-\gamma X_t^\pi)\bar{J}(t))$ can be easily proven to be a submartingale.

Step 3: Note now that the process $(\bar{J}(t))$ is a submartingale not necessarily càd-làg. However, by a theorem of Dellacherie-Meyer [46] (see VI.18), we know that the nonincreasing limit of a sequence of càd-làg submartingales is indistinguishable from a càd-làg adapted process. Hence, there exists a càd-làg version of $(\bar{J}(t))$ which will be denoted by $(\bar{J}_t)$. Note that $(\bar{J}_t)$ is still a submartingale.

Step 4: Let us show that $\bar{J}_t \leq J_t$, $\forall\, t \in [0,T]$ a.s. Since by steps 1 and 3, $(\bar{J}_t)$ is a càd-làg submartingale of class D, it admits the following Doob-Meyer decomposition

$$d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t + d\bar{A}_t,$$

where $\bar{Z} \in L^2(W)$, $\bar{U} \in L^2(M)$ and $(\bar{A}_t)$ is a nondecreasing $\mathbb{G}$-predictable process with $\bar{A}_0 = 0$.
As before, we use the fact that the process $(\exp(-\gamma X_t^\pi)\bar{J}_t)$ is a submartingale for any bounded strategy $\pi \in \mathcal{A}$ to give some necessary conditions satisfied by the process $(\bar{A}_t)$.
Let $\pi \in \mathcal{A}$ be a uniformly bounded strategy. The product rule gives

$$d\big(e^{-\gamma X_t^\pi}\bar{J}_t\big) = d\bar{M}_t^\pi + d\bar{A}_t^\pi,$$

with $\bar{A}_0^\pi = 0$ and

$$\begin{cases} d\bar{A}_t^\pi = e^{-\gamma X_t^\pi}\Big\{d\bar{A}_t + \Big[\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t(e^{-\gamma\pi_t\beta_t}-1)(\bar{U}_t+\bar{J}_t) - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t)\Big]dt\Big\}, \\ d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi}\left[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t)dW_t + \big(\bar{U}_t + \big(e^{-\gamma\pi_t\beta_t}-1\big)(\bar{U}_t + \bar{J}_{t^-})\big)dM_t\right]. \end{cases}$$

Let $\bar{\mathcal{A}}$ be the set of uniformly bounded admissible strategies. Since the process $(e^{-\gamma X_t^\pi}\bar{J}_t)$ is a submartingale for any $\pi \in \bar{\mathcal{A}}$, we have $d\bar{A}_t^\pi \geq 0$ a.s. for any $\pi \in \bar{\mathcal{A}}$. Hence, there exists a process $\bar{K} \in \mathcal{A}^2$ such that

$$d\bar{A}_t = -\operatorname*{ess\,inf}_{\pi\in\bar{\mathcal{A}}}\Big\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}dt + d\bar{K}_t.$$

Now, the following equality holds $dt \otimes d\mathbb{P} - a.e.$ (see Appendix 1.10.5 for details)

$$\operatorname*{ess\,inf}_{\pi \in \bar{\mathcal{A}}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t - \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(\bar{J}_t + \bar{U}_t) \Big\} =$$
$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t - \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(\bar{J}_t + \bar{U}_t) \Big\}. \quad (1.5.2)$$

Hence, $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ is a subsolution of BSDE (1.4.7) and Theorem 1.4.1 implies that

$$\bar{J}_t \leq J_t, \ \forall \, t \in [0, T] \ a.s.,$$

which ends the proof. □

In the next section, we will see that in the classical case of *bounded coefficients*, for each $k \in \mathbb{N}$, (J_t^k) can be characterized as the solution of a *Lipschitz BSDE*.

1.6 Case of bounded coefficients

In this section, the coefficients of the model (μ_t), (σ_t), (β_t) and (λ_t) are supposed to be bounded. We will see that in this case, the two previous theorems (Theorem 1.4.1 and Theorem 1.5.1) will lead to more precise characterizations of the dynamic value function.

For each $k \in \mathbb{N}$, we define the set $\mathcal{B}^k$ as the set of all strategies (not necessarily in $\mathcal{A}$) such that they take their values in $[-k, k]$. Also, we denote by $\mathcal{B}_t^k$ the set of all strategies beginning at t and such that they take their values in $[-k, k]$.

Note that for each $k \in \mathbb{N}$, $\forall \, p > 1$ and $\forall \, t \in [0, T]$ the following integrability property

$$\sup_{\pi \in \mathcal{B}^k} \mathbb{E}\big[\exp\big(-\gamma p X_t^{\pi} \big) \big] < \infty \quad (1.6.1)$$

clearly holds.

We state the following lemma:

Lemma 1.6.1. *The following equality holds for any $k \in \mathbb{N}$ and for any $t \in [0, T]$*

$$J_t^k = \operatorname*{ess\,inf}_{\pi \in \mathcal{B}_t^k} \mathbb{E}\big[\exp\big(-\gamma \big(X_T^{t,\pi} + \xi \big) \big) \big| \mathcal{G}_t \big] \ a.s.,$$

with (J_t^k) defined in the previous section by (1.5.1).

Proof. Fix $k \in \mathbb{N}$ and $t \in [0, T]$. Put $\bar{J}_t^k := \operatorname{ess\,inf}_{\pi \in \mathcal{B}_t^k} \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi)) | \mathcal{G}_t]$. Since $\mathcal{A}_t^k \subset \mathcal{B}_t^k$, we get $\bar{J}_t^k \leq J_t^k$. To prove the other inequality, we state that there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{A}_t^k$ such that $\pi^n \to \pi$, $dt\otimes d\mathbb{P}$ a.s., for any $\pi\in\mathcal{B}_t^k$. Let us define π^n by

$$\pi_s^n = \pi_s \mathbb{1}_{s\le\tau_n},\ \forall\ s\in[t,T],$$

where τ_n is the stopping time defined by $\tau_n = \inf\{s\ge t, |X_s^{t,\pi}|\ge n\}$.
It is clear that for each $n\in\mathbb{N}$, $\pi^n\in\mathcal{A}_t^k$. Thus, $\exp(-\gamma X_T^{t,\pi^n}) = \exp(-\gamma X_{T\wedge\tau_n}^{t,\pi}) \xrightarrow{a.s.} \exp(-\gamma X_T^{t,\pi})$ as $n\to+\infty$. By (1.6.1), the set of random variables $\{\exp(-\gamma X_T^{t,\pi}),\ \pi\in\mathcal{B}_t^k\}$ is uniformly integrable. Hence, $\mathbb{E}[\exp(-\gamma(X_T^{t,\pi^n}+\xi))|\mathcal{G}_t] \to \mathbb{E}[\exp(-\gamma(X_T^{t,\pi}+\xi))|\mathcal{G}_t]$ a.s. as $n\to+\infty$. Therefore, we have $J_t^k \le \bar{J}_t^k$ a.s. which ends the proof. □

Now by Proposition 1.3.1, we have that for each $k\in\mathbb{N}$, the process (J_t^k) is characterized as the solution of a *Lipschitz* BSDE given by (1.3.3) with $\mathcal{C}$ replaced by $\mathcal{B}^k$. Hence, we have that:

Theorem 1.6.1. *(Approximation of the value function)*
The value function is characterized as the nonincreasing limit *of the sequence* $(J_t^k)_{k\in\mathbb{N}}$ *as* k *tends to* $+\infty$*, where for each* k*,* (J_t^k) *is the solution of* Lipschitz BSDE *(1.3.3) with* $\mathcal{C}=\mathcal{B}^k$.

Remark 1.6.1. Note that this allows to *approximate* the value function by *numerical computations* (by applying for example Bouchard and Elie's results [22]).

We now recall a result of convergence stated by Morlais [99]. For each $k\in\mathbb{N}$, let us denote by (Z_t^k, U_t^k) the pair of square integrable processes such that (J^k, Z^k, U^k) is solution of the associated *Lipschitz* BSDE (1.3.3) with $\mathcal{C}$ replaced by $\mathcal{B}^k$. We make the following change of variables

$$\begin{cases} y_t^k &= \dfrac{1}{\gamma}\log(J_t^k),\\ z_t^k &= \dfrac{1}{\gamma}\dfrac{Z_t^k}{J_t^k},\\ u_t^k &= \dfrac{1}{\gamma}\log\Big(1+\dfrac{U_t^k}{J_{t^-}^k}\Big). \end{cases}$$

It is clear that the process (y^k, z^k, u^k) is a solution of the following quadratic BSDE

$$-\,dy_t^k = g^k(t, z_t^k, u_t^k)dt - z_t^k dW_t - u_t^k dM_t\ ;\ y_T^k = -\xi\ ,$$

with

$$g^k(s,z,u) = \inf_{\pi\in\mathcal{B}^k}\Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z+\frac{\mu_s+\lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u-\pi_s\beta_s|_\gamma\Big) - (\mu_s+\lambda_s\beta_s)z - \frac{|\mu_s+\lambda_s\beta_s|^2}{2\gamma}$$

and $|u-\pi\beta_t|_\gamma = \lambda_t\frac{\exp(\gamma(u-\pi\beta_t))-1-\gamma(u-\pi\beta_t)}{\gamma}$.
Recall that by using Kobylanski's techniques [83] on monotone stability convergence theorem, Morlais [99] shows the following nice result:

Proposition 1.6.1. *(Morlais's result) Suppose that the coefficients are bounded and that* ξ *is bounded. Then,* (y_t^k, z_t^k, u_t^k) *converges to* (y_t, z_t, u_t) *in the following sense*

$$\mathbb{E}(\sup_{t\in[0,T]} |y_t^k - y_t|) + |z^k - z|_{L^2(W)} + |u^k - u|_{L^2(M)} \to 0 \ ,$$

where (y, z, u) *is solution of*

$$-\, dy_t = g(t, y_t, z_t, u_t)dt - z_t dW_t - u_t dM_t \ ; \ y_T = -\xi \ ,$$

with

$$g(s, z, u) = \inf_{\pi\in\bar{\mathcal{B}}} \Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s + \lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u - \pi_s\beta_s|_\gamma\Big) - (\mu_s + \lambda_s\beta_s)z - \frac{|\mu_s + \lambda_s\beta_s|^2}{2\gamma},$$

and $\bar{\mathcal{B}} = \cup_k \mathcal{B}^k$.

By similar arguments as in the proof of the above lemma (Lemma 1.6.1) or as in Appendix 1.10.5, the set $\bar{\mathcal{B}}$ can be replaced by $\bar{\mathcal{A}}$ or even by $\mathcal{A}$.

Using this proposition and our characterization of (J_t) as the *nonincreasing limit* of $((J_t^k))_{k\in\mathbb{N}}$, we can identify the limit (y_t). More precisely, let us define the following processes

$$\begin{cases} J_t^* = e^{\gamma y_t}, \\ Z_t^* = \gamma J_t^* z_t, \\ U_t^* = (e^{\gamma u_t} - 1)J_{t^-}^*. \end{cases}$$

Since $J_t = \lim_{k\to\infty} J_t^k$ by Theorem 1.6.1 (or 1.5.1), $J_t^* = J_t$, $\forall\, t \in [0,T]$ a.s., and the uniqueness of the Doob-Meyer decomposition (1.4.6) of J_t implies that $Z_t^* = Z_t$ and $U_t^* = U_t$ $dt \otimes d\mathbb{P} - a.e.$ Also, by using Morlais's result (Proposition 1.6.1), we derive that (J, Z, U) is a solution to BSDE (1.4.7), and not only a subsolution. This, with the characterization of (J_t) of Theorem 1.4.1, give:

Theorem 1.6.2. *(Characterization of the value function)*
Suppose that ξ *and the coefficients are bounded. Then, the value function* (J, Z, U) *is the maximal solution of BSDE (1.4.7) (that is with* $K_t = 0$ *for any* $t \in [0,T]$*).*

Remark 1.6.2. Moreover, if there is no default, our result corresponds to that of Hu *et al.* [67] in the complete case (by making the simple exponential change of variable $y_t = \frac{1}{\gamma}\log(J_t)$). Also, in this case, the optimal strategy belongs to the set $\mathcal{A}'$. Indeed, the optimal terminal wealth is given by $\hat{X}_T = I(\lambda Z_0(T))$, where I is the inverse of U', λ is a fixed parameter, $Z_0(T) := \exp\{-\int_0^T \alpha_t dW_t - \frac{1}{2}\int_0^T \alpha_t^2 dt\}$ and $\alpha_t := \frac{\mu_t+\lambda_t\beta_t}{\sigma_t}$ (supposed to be bounded).

1.7 Case of coefficients which satisfy some exponential integrability conditions

In this section, we will study the case of coefficients not necessarily bounded but satisfying some integrability conditions. We will first study the particular case of strategies valued in a convex-compact set. Then, we will see that the approximation result of the value function in the non constrained case (Theorem 1.5.1) can be specified.

1.7.1 Case of strategies valued in a convex-compact set

We suppose that the set of admissible strategies is given by $\mathcal{C}$ (see Section 1.3) where C is a convex-compact (not only compact) set. Here, it simply corresponds to a closed interval of $\mathbb{R}$ because we are in the one dimensional case. However, the following results clearly still hold in the multidimensional case (see Section 1.9). Let $(J(t))$ be the associated dynamic value function with $\mathcal{C}_t$ defined as in Section 1.3 (see (1.3.1)). Using some classical results of convex analysis (see for example Ekeland and Temam [52]), we easily state the following existence property:

Proposition 1.7.1. *There exists an optimal strategy $\hat{\pi} \in \mathcal{C}$ for the optimization problem (1.2.5), that is*

$$J(0) = \inf_{\pi \in \mathcal{C}} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^\pi + \xi\big)\big)\big] = \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{\hat{\pi}} + \xi\big)\big)\big].$$

Proof. Note that $\mathcal{C}$ is strongly closed and convex in $L^2([0,T] \times \Omega)$. Hence, $\mathcal{C}$ is closed for the weak topology. Moreover, since $\mathcal{C}$ is bounded, $\mathcal{C}$ is compact for the weak topology.
We define the function $\phi(\pi) = \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$ on $L^2([0,T] \times \Omega)$. This function is clearly convex and continuous for the strong topology in $L^2([0,T] \times \Omega)$. By classical results of convex analysis, it is s.c.i for the weak topology. Now, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{C}$ such that $\phi(\pi^n) \to \min_{\pi\in\mathcal{C}} \phi(\pi)$ as $n \to \infty$.
Since $\mathcal{C}$ is weakly compact, there exists an extracted sequence still denoted by (π^n) which converges for the weak topology to $\hat{\pi}$ for some $\hat{\pi} \in \mathcal{C}$. Now, since ϕ is s.c.i for the weak topology, it implies that

$$\phi(\hat{\pi}) \leq \liminf \phi(\pi^n) = \min_{\pi\in\mathcal{C}} \phi(\pi).$$

Therefore, $\phi(\hat{\pi}) = \inf_{\pi\in\mathcal{C}} \phi(\pi)$ and the proof is ended. □

We now want to characterize the value function $J(t)$ by a BSDE. For that, we cannot apply the same techniques as in the case of bounded coefficients. Indeed, since the coefficients are not necessarily bounded, the drivers of the associated BSDEs are no longer

Lipschitz. Hence, the existence and uniqueness results and also the comparison theorem do not a priori hold. Therefore, as in Section 1.4, we will use dynamic programming techniques of stochastic control but also the existence of an optimal strategy.

First, one can show easily that the set $\{J_t^\pi,\ \pi \in \mathcal{C}_t\}$ is *stable by pairwise minimization.* In order to have the dynamic programming principle, we now suppose that the coefficients satisfy the following integrability condition:

Assumption 1.7.1. (β_t) *is uniformly bounded and*

$$\mathbb{E}\Big[\exp\Big(a\int_0^T |\mu_t| dt\Big)\Big] + \mathbb{E}\Big[\exp\Big(b\int_0^T |\sigma_t|^2 dt\Big)\Big] < \infty,$$

where $a = 2\gamma||\mathcal{C}||_\infty$ *and* $b = 8\gamma^2||\mathcal{C}||_\infty^2$.

By classical computations, one can easily derive that for any $t \in [0,T]$ and any $\pi \in \mathcal{C}_t$, the following inequality holds

$$\mathbb{E}\big[\sup_{s\in[t,T]} \exp\big(-\gamma X_s^{t,\pi}\big)\big] < \infty. \tag{1.7.1}$$

Using this integrability property and similar arguments as in the proof of Proposition 1.4.1, the process $(J(t))$ can be shown to satisfy the *dynamic programming principle* over $\mathcal{C}$ that is: $(J(t))$ is the largest $\mathbb{G}$-adapted process such that $(\exp(-\gamma X_t^\pi)J(t))$ is a submartingale for any $\pi \in \mathcal{C}$ with $J(T) = \exp(-\gamma\xi)$.

Also, the following classical optimality criterion holds:

Proposition 1.7.2. *Let* $\hat{\pi} \in \mathcal{C}$. *The two following assertions are equivalent:*

(i) $\hat{\pi} \in \mathcal{C}$ *is optimal that is* $J(0) = \mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))]$.

(ii) The process $(\exp(-\gamma X_t^{\hat{\pi}})J(t))$ *is a martingale.*

The proof is given in Appendix 1.10.6.

Corollary 1.7.1. *There exists a càd-làg version of* $(J(t))$ *which will be denoted by* (J_t).

Proof. The proof is simple here because we have an existence result. More precisely, from Proposition 1.7.1, there exists $\hat{\pi} \in \mathcal{C}$ which is optimal for J_0. Hence, by the optimality criterium (Proposition 1.7.2), we have $J(t) = \exp(-\gamma X_t^{\hat{\pi}})\mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))|\mathcal{G}_t]$ for any $t \in [0,T]$ (in other words, $\hat{\pi}$ is also optimal for $J(t)$). By classical results on the conditional expectation, there exists a càd-làg version denoted by (J_t). $\square$

Note that the process (J_t) verifies $0 \leq J_t \leq 1$, $\forall t \in [0,T]$ a.s. Using the dynamic programming principle and the existence of an optimal strategy, we state the following property:

Proposition 1.7.3. *There exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J,Z,U) is the maximal solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (1.3.3).*

The proof is given in Appendix 1.10.7.

Remark 1.7.1. It can be noted that the optimal strategy $\hat{\pi} \in \mathcal{C}$ for J_0 is characterized by the fact that $\hat{\pi}_t$ attains the essential infimum in (1.3.3), $dt \otimes d\mathbb{P} - a.e.$

With Assumption 1.7.1 it is possible to prove the unicity of the solution to BSDE (1.3.3).

Theorem 1.7.1. *(Characterization of the value function)*
The value function (J,Z,U) is characterized as the unique solution *in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (1.3.3).*

Proof. Let $(\bar{J},\bar{Z},\bar{U})$ be a solution of BSDE (1.3.3). Using a measurable selection theorem, we know that there exists at least a strategy $\bar{\pi} \in \mathcal{C}$ such that $dt \otimes d\mathbb{P} - a.e.$

$$\begin{aligned}\operatorname*{ess\,inf}_{\pi\in\mathcal{C}} &\Big\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}\\ &= \frac{\gamma^2}{2}\bar{\pi}^2\sigma_t^2\bar{J}_t - \gamma\bar{\pi}_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\bar{\pi}_t\beta_t}\big)(\bar{J}_t+\bar{U}_t).\end{aligned}$$

Thus (1.3.3) can be written under the form

$$d\bar{J}_t = \Big\{\gamma\bar{\pi}_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \lambda_t\big(1-e^{-\gamma\bar{\pi}_t\beta_t}\big)(\bar{J}_t+\bar{U}_t) - \frac{\gamma^2}{2}\bar{\pi}^2\sigma_t^2\bar{J}_t\Big\}dt + \bar{Z}_t dW_t + \bar{U}_t dM_t.$$

Let us introduce by $B_t = \exp(-\gamma X_t^{\bar{\pi}})$. Itô's formula and rule product give

$$d(B_t\bar{J}_t) = \big(B_t\bar{Z}_t - \gamma\sigma_t\bar{\pi}_t B_t\bar{J}_t\big)dW_t + \big[(e^{-\gamma\beta_t\bar{\pi}_t}-1)B_{t^-}\bar{J}_t + e^{-\gamma\beta_t\bar{\pi}_t}B_{t^-}\bar{U}_t\big]dM_t.$$

By Assumption 1.7.1 and since $(\bar{J}_t)$ is bounded, one can derive that the local martingale $(B_t\bar{J}_t)$ satisfies $\mathbb{E}[\sup_{0\leq t\leq T}|B_t\bar{J}_t|] < \infty$. Hence, $(B_t\bar{J}_t)$ is a martingale. Thus,

$$\bar{J}_t = \mathbb{E}\Big[\frac{B_T}{B_t}e^{-\gamma\xi}\Big|\mathcal{G}_t\Big] = \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\bar{\pi}}+\xi\big)\big)\big|\mathcal{G}_t\big].$$

Thus,

$$\bar{J}_t \geq \operatorname*{ess\,inf}_{\pi\in\mathcal{C}}\mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi}+\xi\big)\big)\big|\mathcal{G}_t\big] = J_t.$$

Now, by the previous Proposition 1.7.3, (J_t) is the maximal solution of BSDE (1.3.3). This gives that for any $t \in [0,T]$, $J_t \leq \bar{J}_t$, $\mathbb{P}-a.s.$ Hence, $J_t = \bar{J}_t$, $\forall\, t \in [0,T]$, $\mathbb{P}-a.s.$, and $\bar{\pi}$ is optimal and the proof is ended.

□

1.7.2 The non constrained case

In this section, the set of admissible strategies is given by $\mathcal{A}$. Under some exponential integrability conditions on the coefficients, we can also precise the characterization of the value function (J_t) as the limit of $((J_t^k))_{k\in\mathbb{N}}$ as k tends to $+\infty$.

Assumption 1.7.2. (β_t) is uniformly bounded, $\mathbb{E}[\int_0^T \lambda_t dt] < \infty$ and for any $p > 0$ we have

$$\mathbb{E}\Big[\exp\Big(p\int_0^T |\mu_t| dt\Big)\Big] + \mathbb{E}\Big[\exp\Big(p\int_0^T |\sigma_t|^2 dt\Big)\Big] < \infty.$$

Again, for each $k \in \mathbb{N}$, we consider the set $\mathcal{B}_t^k$ of strategies beginning at t and valued in $[-k,k]$. Since Assumption 1.7.2 is satisfied, the integrability condition (1.6.1) holds and hence, for each $k \in \mathbb{N}$,

$$J_t^k = \operatorname*{ess\,inf}_{\pi\in\mathcal{B}_t^k} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\pi}+\xi\big)\big)\big|\mathcal{G}_t\big] \ a.s.$$

In this case, for each $k \in \mathbb{N}$, the process (J_t^k) is characterized as the unique solution of BSDE (1.3.3) with $\mathcal{C} = \mathcal{B}^k$. Therefore, we have:

Theorem 1.7.2. *(Characterization of the value function)*
The value function is characterized as the nonincreasing limit *of the sequence* $((J_t^k))_{k\in\mathbb{N}}$ *as* k *tends to* $+\infty$*, which are* the unique solutions *of BSDEs (1.3.3) with* $\mathcal{C} = \mathcal{B}^k$ *for each* $k \in \mathbb{N}$.

1.8 Indifference pricing via the maximization of exponential utility

We first present a general framework of the Hodges and Neuberger [66] approach with some strictly increasing, strictly concave and continuously differentiable mapping U, defined on $\mathbb{R}$. We solve explicitly the problem in the case of exponential utility.

The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows the issue at hand is to assess the value of some (defaultable) claim ξ as seen from the perspective of an investor who optimizes his behavior relative to some utility function, say U. The investor has two choices

– he only invests in the risk-free asset and in the risky asset, in this case the associated optimization problem is
$$V(x,0) = \sup_{\pi} \mathbb{E}\big[U\big(X_T^{x,\pi}\big)\big],$$

– he also invests in the contingent claim, whose price is p at 0, in this case the associated optimization problem is
$$V(x-p,\xi) = \sup_{\pi} \mathbb{E}\big[U\big(X_T^{x-p,\pi} + \xi\big)\big].$$

Definition 1.8.1. For a given initial endowment x, the Hodges buying price of a defaultable claim ξ is the price p such that the investor's value functions are indifferent between holding and not holding the contingent claim ξ, i.e.
$$V(x,0) = V(x-p,\xi).$$

Remark 1.8.1. We can define the Hodges selling price p_* of ξ by considering $-p$, where p is the buying price of $-\xi$, as specified in the previous definition.

In the rest of this section, we consider the case of an exponential utility function. With our notation, if the investor buys the contingent claim at the price p and invests the rest of his money in the risk-free asset and in the risky asset, the value function is equal to
$$V(x-p,\xi) = \exp(-\gamma(x-p))V(0,\xi).$$
If he invests all his money in the risk-free asset and in the risky asset, the value function is equal to
$$V(x,0) = \exp(-\gamma x)V(0,0).$$
Hence, the Hodges price for the contingent claim ξ is given by the formula
$$p = \frac{1}{\gamma}\ln\Big(\frac{V(0,0)}{V(0,\xi)}\Big) = \frac{1}{\gamma}\ln\Big(\frac{J(0,0)}{J(0,\xi)}\Big).$$
since $J(0,\xi) = -V(0,\xi)$.

In the case of Section 1.3, that is where the strategies take their values in a compact set C, we have:

Proposition 1.8.1. *(Compact case) Suppose that the coefficients are bounded. Let (J_t^ξ) be the solution of Lipschitz BSDE (1.3.3) and (J_t^0) be the solution of Lipschitz BSDE (1.3.3) with $\xi = 0$. The Hodges price for the contingent claim ξ is given by the formula*
$$p = \frac{1}{\gamma}\ln\Big(\frac{J_0^0}{J_0^\xi}\Big). \tag{1.8.1}$$

Remark 1.8.2. Consequently, the indifference price is simply given in terms of two Lipschitz BSDEs. This leads to possible numerical computations by applying the results of Bouchard and Elie [22].
Note that in the case where the coefficients are not supposed to be bounded but only satisfy some exponential integrability conditions (see Section 1.7), Proposition 1.8.1 still holds except that BSDE (1.3.3) is no longer Lipschitz (but still admits a unique solution).

In the non constrained case, without any assumptions on the coefficients, we have

Proposition 1.8.2. *(Non constrained case) Let (J_t^ξ) (resp. (J_t^0)) be the maximal subsolution of BSDE (1.4.7) (resp. with $\xi = 0$). The Hodges price for the contingent claim ξ associated with $\mathcal{A}$ is given by formula (1.8.1).*

Note that if the coefficient β is bounded (but not necessarily the others), the indifference price associated with the set Θ_2 of Delbaen *et al.* [45] and that associated with the set $\mathcal{A}$ coincide because the value functions $V(x,0)$ and $V(x-p,\xi)$ are the same for Θ_2 or $\mathcal{A}$.

Recall that in the case of bounded coefficients, (J_t^ξ) is the maximal solution of BSDE (1.4.7). Also, in this case, we have:

Proposition 1.8.3. *(Approximation of the indifference price) Suppose that the coefficients are bounded. The Hodges price p for the contingent claim ξ associated with Θ_2 (or equivalently with $\mathcal{A}$) satisfies*

$$p = \lim_{k \to \infty} p^k,$$

where for each k, p^k is the Hodges price associated with the simple set $\mathcal{B}^k$ of all strategies bounded by k. For each k, p^k is given by

$$p^k = \frac{1}{\gamma} \ln \Big(\frac{J_0^{k,0}}{J_0^{k,\xi}} \Big),$$

where $(J_t^{k,\xi})$ (resp. $(J_t^{k,0})$) is the solution of Lipschitz BSDE (1.3.3) (resp. with $\xi = 0$) with $\mathcal{C} = \mathcal{B}^k$.

Remark 1.8.3. This leads to possible numerical computations in order to approximate the indifference price. Also, note that in the case where the coefficients are not supposed to be bounded but only satisfy some exponential integrability conditions (see Section 1.7), Proposition 1.8.3 still holds except that BSDE (1.3.3) is no longer Lipschitz (but still admits a unique solution).

1.9 Generalizations

In this section, we give some generalizations of the previous results. The proofs are not given, but they are identical to the proofs of the case with a default time and a stock. In all this section, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript $'$ stands for the transposition, $||.||$ the square norm, $\mathbb{1}$ the vector of $\mathbb{R}^n$ such that each component of this vector is equal to 1. Let U and V be two vectors of $\mathbb{R}^n$, $U * V$ denotes the vector such that $(U * V)_i = U_i V_i$ for each $i \in \{1, \dots, n\}$. Let $X \in \mathbb{R}^n$, $diag(X)$ is the matrix such that $diag(X)_{ij} = X_i$ if $i = j$ else $diag(X)_{ij} = 0$.

1.9.1 Several default times and several stocks

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two stochastic processes: an n-dimensional Brownian motion (W_t) and a p-dimensional jump process $(N_t) = ((N_t^i), 1 \leq i \leq p)$ with $N_t^i = \mathbb{1}_{\tau^i \leq t}$, where $(\tau^i)_{1 \leq i \leq p}$ are p default times. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes.

Assumption 1.9.1. We make the following assumptions on the default times:

(i) The defaults do not appear simultaneously: $\mathbb{P}(\tau^i = \tau^j) = 0$ for $i \neq j$.

(ii) Each default can appear at any time: $\mathbb{P}(\tau^i > t) > 0$.

We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at any time, and n risky assets, whose price processes $(S_t^i)_{1 \leq i \leq n}$ admit p discontinuities at times $(\tau^j)_{1 \leq j \leq p}$. In the sequel, we consider that the price processes $(S_t^i)_{1 \leq i \leq n}$ evolve according to the equation

$$dS_t = diag(S_{t^-})(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \tag{1.9.1}$$

with the classical assumptions:

Assumption 1.9.2.

(i) (μ_t), (σ_t) and (β_t) are $\mathbb{G}$-predictable processes such that σ_t is nonsingular for any $t \in [0, T]$ and

$$\int_0^T ||\sigma_t||^2 dt + \sum_{i,j} \int_0^T \lambda_t^j |\beta_t^{i,j}|^2 dt < \infty \ a.s.,$$

(ii) there exist d coefficients $\theta^1, \dots, \theta^d$ that are $\mathbb{G}$-predictable processes such that

$$\mu_t^i + \sum_{j=1}^p \lambda_t^j \beta_t^{i,j} = \sum_{j=1}^d \sigma_t^{i,j} \theta_t^j, \ \forall\, t \in [0, T] \ a.s., \ 1 \leq i \leq n;$$

we suppose that θ^j is bounded,

(iii) the processes $(\beta_t^{i,j})$ satisfy $\beta_{\tau_j}^{i,j} > -1$ a.s., for each i and j.

Using the same techniques as in the previous sections, we can generalize all the results stated in the previous sections to this framework. In particular, in the classical case of bounded coefficients, if (J_t) denotes the dynamic value function associated with the admissible sets $\mathcal{A}$ or $\mathcal{A}'$ which are equal, we have:

Theorem 1.9.1. *There exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is the maximal solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the BSDE*

$$\begin{cases} -\, dJ_t = & \underset{\pi \in \mathcal{A}}{\operatorname{ess\,inf}} \Big\{ \dfrac{\gamma^2}{2} ||\pi_t' \sigma_t||^2 J_t - \gamma \pi_t' (\mu_t J_t + \sigma_t Z_t) - (\mathbb{1} - e^{-\gamma \pi_t' \beta_t})(\lambda_t J_t + \lambda_t * U_t) \Big\} dt \\ & -\; Z_t dW_t - U_t dM_t, \\ J_T = & \exp(-\gamma \xi). \end{cases}$$

Remark 1.9.1. The value function J_0 coincides with the value function associated with the set Θ_2.

1.9.2 Poisson jumps

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two independent processes: a unidimensional Brownian motion (W_t) and a real-valued Poisson point process p defined on $[0, T] \times \mathbb{R}\backslash\{0\}$, we denote by $N_p(ds, dx)$ the associated counting measure, such that its compensator is $\hat{N}_p(ds, dx) = n(dx)ds$, and the Levy measure $n(dx)$ is positive and satisfies $n(\{0\}) = 0$ and $\int_{\mathbb{R}\backslash\{0\}} (1 \wedge |x|)^2 n(dx) < \infty$. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by the two processes (W_t) and (N_p). We denote by $\tilde{N}_p(ds, dx)$ ($\tilde{N}_p(ds, dx) = N_p(ds, dx) - \hat{N}_p(ds, dx)$) the compensated measure, which is a martingale random measure. In particular, for any predictable and locally square integrable process (U_t), the stochastic integral $\int U_s(x) \tilde{N}_p(ds, dx)$ is a locally square integrable martingale. Let us introduce the classical set $L^2(\tilde{N}_p)$ (resp. $L^2_{loc}(\tilde{N}_p)$) given by the set of $\mathbb{G}$-predictable processes on $[0, T]$ under $\mathbb{P}$ with

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}\backslash\{0\}} |U_t(x)|^2 n(dx) dt\Big] < \infty \text{ (resp. } \int_0^T \int_{\mathbb{R}\backslash\{0\}} |U_t(x)|^2 n(dx) dt < \infty \ a.s.).$$

The financial market consists of one risk-free asset, whose price process is assumed to be equal to 1, and one single risky asset, whose price process is denoted by S. In particular, the stock price process satisfies

$$dS_t = S_{t^-} \Big(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\backslash\{0\}} \beta_t(x) N_p(dt, dx) \Big).$$

All processes (μ_t), (σ_t) and (β_t) are assumed to be $\mathbb{G}$-predictable, the process (σ_t) satisfies $\sigma_t > 0$ and the process (β_t) satisfies $\beta_t(x) > -1$ a.s. Moreover we suppose that

$$\int_0^T |\sigma_t|^2 dt + \int_0^t \int_{\mathbb{R}\setminus\{0\}} |\beta_t(x)|^2 n(dx)dt + \int_0^T \Big|\frac{\mu_t + \int_{\mathbb{R}\setminus\{0\}} \beta_t(x) n(dx)ds}{\sigma_t}\Big|^2 dt < \infty \ a.s.$$

Using the same techniques as in the previous sections, we can generalize all the results stated in the previous sections to this framework. In particular, in the classical case of bounded coefficients, if (J_t) denotes the dynamic value function associated with the admissible sets $\mathcal{A}$ or $\mathcal{A}'$ which are equal, we have:

Theorem 1.9.2. *There exist $Z \in L^2(W)$ and $U \in L^2(\tilde{N}_p)$ such that (J, Z, U) is the maximal solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(\tilde{N}_p)$ of the BSDE*

$$\begin{cases} -\, dJ_t = \underset{\pi\in\mathcal{A}}{\operatorname{ess\,inf}} \Big\{ \dfrac{\gamma^2}{2} |\pi_t \sigma_t|^2 J_t - \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) - \displaystyle\int_{\mathbb{R}\setminus\{0\}} (1 - e^{-\gamma\pi_t x})(J_t + U_t(x)) n(dx) \Big\} dt \\ \qquad\quad -\, Z_t dW_t - \displaystyle\int_{\mathbb{R}\setminus\{0\}} U_t(x) \tilde{N}_p(dt, dx)\,, \\ J_T = \exp(-\gamma\xi)\,. \end{cases}$$

Remark 1.9.2. The value function J_0 coincides with the value function associated with the set Θ_2.

1.10 Appendix

1.10.1 Essential supremum

Recall the following classical result (see Neveu [100]):

Theorem 1.10.1. *Let F be a non empty family of measurable real valued functions $f : \Omega \to \bar{\mathbb{R}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a measurable function $g : \Omega \to \bar{\mathbb{R}}$ such that*

(i) for all $f \in F$, $f \leq g$ a.s.,

(ii) if h is a measurable function satisfying $f \leq h$ a.s., for all $f \in F$, then $g \leq h$ a.s.

This function g, which is unique a.s., is called the essential supremum of F and is denoted $\operatorname{ess\,sup}_{f\in F} f$.

Moreover there exists at least one sequence (f_n) in F such that $\operatorname{ess\,sup}_{f\in F} f = \lim_{n\to\infty} f_n$ *a.s. Furthermore, if F is* filtrante croissante *(i.e. f, $g \in F$ then there exists $h \in F$ such* *that both $f \leq h$ a.s., and $g \leq h$ a.s.), then the sequence (f_n) may be taken nondecreasing and* $\operatorname{ess\,sup}_{f \in F} f = \lim_{n\to\infty} \uparrow f_n$ *a.s.*

1.10.2 A classical lemma of analysis

Lemma 1.10.1. *The supremum of affine functions, whose coefficients are bounded by a constant $c > 0$, is Lipschitz and the Lipschitz constant is equal to c.*
More precisely, let $\mathcal{A}$ be the set of $[-c,c]^n \times [-k,k]$. Then, the function f defined for any $y \in \mathbb{R}^n$ by

$$f(y) = \sup_{(a,b)\in\mathcal{A}} \{a.y + b\}$$

is Lipschitz with Lipschitz constant c.

Proof.

$$\sup_{(a,b)\in\mathcal{A}} \{a.y + b\} \leq \sup_{(a,b)\in\mathcal{A}} \{a.(y - y')\} + \sup_{(a,b)\in\mathcal{A}} \{a.y' + b\}.$$

Which implies

$$f(y) - f(y') \leq c||y - y'||.$$

By symmetry, we have also

$$f(y') - f(y) \leq c||y - y'||,$$

which gives the desired result. □

1.10.3 Proof of the closedness by binding of $\mathcal{A}'$

Lemma 1.10.2. *Let π^1, π^2 be two admissible strategies of $\mathcal{A}'$ and $s \in [0,T]$. The strategy π^3 defined by*

$$\pi^3_t = \begin{cases} \pi^1_t & \text{if } t \leq s, \\ \pi^2_t & \text{if } t > s, \end{cases}$$

belongs to $\mathcal{A}'$.

Proof. For any $u \in [0,T]$, we have for any $p > 1$

(i) if $u > s$, then

$$\mathbb{E}\big[\sup_{r\in[u,T]} \exp\big(-\gamma p X^{u,\pi^3}_r\big)\big] = \mathbb{E}\big[\sup_{r\in[u,T]} \exp\big(-\gamma p X^{u,\pi^2}_r\big)\big] < \infty,$$

(ii) if $u \leq s$, then

$$\mathbb{E}\big[\sup_{r\in[u,T]} \exp\big(-\gamma p X_r^{u,\pi^3}\big)\big] \leq \mathbb{E}\big[\sup_{r\in[u,T]} \exp\big(-\gamma p X_r^{u,\pi^1}\big)\big] + \mathbb{E}\big[\sup_{r\in[s,T]} \exp\big(-\gamma p\big(X_s^{u,\pi^1} + X_r^{s,\pi^2}\big)\big)\big].$$

By Cauchy-Schwarz inequality,

$$\mathbb{E}\big[\sup_{r\in[s,T]} \exp\big(-\gamma p\big(X_s^{u,\pi^1} + X_r^{s,\pi^2}\big)\big)\big] \leq \mathbb{E}\big[\sup_{r\in[u,T]} \exp\big(-2\gamma p X_r^{u,\pi^1}\big)\big]^{1/2} \times \mathbb{E}\big[\sup_{r\in[s,T]} \exp\big(-2\gamma p X_r^{s,\pi^2}\big)\big]^{1/2}.$$

Hence, $\mathbb{E}[\sup_{r\in[u,T]} \exp(-\gamma p X_r^{u,\pi^3})] < \infty$.

□

1.10.4 Proof of the existence of a càd-làg modification of (J_t)

The proof is not so simple since we do not know if there exists an optimal strategy in $\mathcal{A}$. Let $\mathbb{D} = [0,T] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Since $(J(t))$ is a submartingale, the mapping $t \to J(t,\omega)$ defined on $\mathbb{D}$ has for almost every $\omega \in \Omega$ and for any t of $[0,T[$ a finite right limit

$$J(t^+,\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} J(s,\omega),$$

(see Karatzas and Shreve [79], Proposition 1.3.14 or Dellacherie and Meyer [46], Chapter 6). Note that it is possible to define $J(t^+,\omega)$ for any $(t,\omega) \in [0,T] \times \Omega$ by $J(T^+,\omega) := J(T,\omega)$ and

$$J(t^+,\omega) := \limsup_{s\in\mathbb{D}, s\downarrow t} J(s,\omega),\ \ t \in [0,T[\,.$$

From the right-continuity of the filtration $(\mathcal{G}_t)$, the process $(J(t^+))$ is $\mathbb{G}$-adapted. It is possible to show that $(J(t^+))$ is a $\mathbb{G}$-submartingale and even that the process $(\exp(-\gamma X_t^\pi) J(t^+))$ is a $\mathbb{G}$-submartingale for any $\pi \in \mathcal{A}$. Indeed, from Proposition 1.4.2, for any $s \leq t$ and for each sequence of rational numbers $(t_n)_{n\geq 1}$ converging down to t, we have

$$\mathbb{E}\big[\exp\big(-\gamma X_{t_n}^\pi\big) J(t_n) \big| \mathcal{G}_s\big] \geq \exp\big(-\gamma X_s^\pi\big) J(s)\ a.s.$$

Let n tend to $+\infty$. By the Lebesgue theorem, we have that for any $s \leq t$,

$$\mathbb{E}\big[\exp\big(-\gamma X_t^\pi\big) J(t^+) \big| \mathcal{G}_s\big] \geq \exp\big(-\gamma X_s^\pi\big) J(s)\ a.s. \tag{1.10.1}$$

This clearly implies that for any $s \leq t$, $\mathbb{E}[\exp(-\gamma X_t^\pi) J(t^+)|\mathcal{G}_s] \geq \exp(-\gamma X_s^\pi) J(s^+)$ a.s., which gives the submartingale property of the process $(\exp(-\gamma X_t^\pi) J(t^+))$. Using the right-continuity of the filtration $(\mathcal{G}_t)$ and inequality (1.10.1) applied to $\pi = 0$ and $s = t$, we get

$$J(t^+) = \mathbb{E}\big[J(t^+)\big|\mathcal{G}_t\big] \geq J(t)\ a.s.$$

On the other hand, by the characterization of $(J(t))$ (see Proposition 1.4.2), and since the process $(\exp(-\gamma X_t^\pi)J(t^+))$ is a $\mathbb{G}$-submartingale for any $\pi \in \mathcal{A}$, we have that for any $t \in [0,T]$,

$$J(t^+) \leq J(t) \ a.s.$$

Thus, for any $t \in [0,T]$,

$$J(t^+) = J(t) \ a.s.$$

Furthermore, the process $(J(t^+))$ is càd-làg. The result follows by taking $J_t = J(t^+)$.

1.10.5 Proof of equality (1.5.2)

For any $\pi \in \mathcal{A}$, we define the strategy $\pi_t^k = \pi_t \mathbb{1}_{|\pi_t| \leq k}$ for each $k \in \mathbb{N}$. The strategy π^k is uniformly bounded but not necessarily admissible. For that we define for each $(k,n) \in \mathbb{N} \times \mathbb{N}$ the stopping time

$$\tau_{k,n} := \inf\{t, |X_t^{\pi^k}| \geq n\}$$

and the strategy $\pi_t^{k,n} := \pi_t^k \mathbb{1}_{t \leq \tau_{k,n}}$. By construction, it is clear that the strategy $\pi^{k,n} \in \mathcal{A}^k$ for each (k,n). Since $\pi_t = \lim_k \lim_n \pi_t^{k,n}$ $dt \otimes d\mathbb{P}$ a.s., the following equality

$$\operatorname*{ess\,inf}_{\pi \in \bar{\mathcal{A}}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t - \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(\bar{J}_t + \bar{U}_t) \Big\} =$$
$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t \gamma \pi_t - (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(\bar{J}_t + \bar{U}_t) \Big\}$$

holds $dt \otimes d\mathbb{P}$ a.s.

1.10.6 Proof of optimality criterion (Proposition 1.7.2)

Suppose (i). Hence,

$$J(0) = \inf_{\pi \in \mathcal{A}} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^\pi + \xi\big)\big)\big] = \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{\hat{\pi}} + \xi\big)\big)\big].$$

As the process $(\exp(-\gamma X_t^{\hat{\pi}})J(t))$ is a submartingale and as $J(0) = \mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))]$, it follows that $(\exp(-\gamma X_t^{\hat{\pi}})J(t))$ is a martingale.
To show the converse, suppose that the process $(\exp(-\gamma X_t^{\hat{\pi}})J(t))$ is a martingale. Then, $\mathbb{E}[\exp(-\gamma X_T^{\hat{\pi}})J(T)] = J(0)$. Also, since the process $(\exp(-\gamma X_t^\pi)J(t))$ is a submartingale for any $\pi \in \mathcal{A}$ and since $J(T) = \exp(-\gamma\xi)$, it is clear that $J(0) \leq \inf_{\pi \in \mathcal{A}} \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$. Consequently,

$$J(0) = \inf_{\pi \in \mathcal{A}} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^\pi + \xi\big)\big)\big] = \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{\hat{\pi}} + \xi\big)\big)\big],$$

thus $\hat{\pi}$ is an optimal strategy.

1.10.7 Characterization of the value function as the maximum solution of BSDE (1.3.3)

Step 1: Let us show that there exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is a solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (1.3.3).

From the Doob-Meyer decomposition, since the process (J_t) is a bounded submartingale, there exist $Z \in L^2(W)$, $U \in L^2(M)$ and (A_t) a nondecreasing process with $A_0 = 0$ such that

$$dJ_t = Z_t dW_t + U_t dM_t + dA_t.$$

By the same techniques as in the proof of Proposition 1.4.4, since for any $\pi \in \mathcal{C}$ the process $(\exp(-\gamma X_t^\pi)J(t))$ is a submartingale, we have

$$dA_t \geq \operatorname*{ess\,sup}_{\pi \in \mathcal{C}} \Big\{ \gamma \pi_t (\mu_t J_t + \sigma_t Z_t) + \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(J_t + U_t) - \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t \Big\} dt.$$

Since there exists an optimal strategy $\hat{\pi} \in \mathcal{C}$ from Proposition 1.7.1, the optimality criterion gives

$$dA_t = \Big\{ \gamma \hat{\pi}_t (\mu_t J_t + \sigma_t Z_t) + \lambda_t \big(1 - e^{-\gamma \hat{\pi}_t \beta_t}\big)(J_t + U_t) - \frac{\gamma^2}{2} \hat{\pi}_t^2 \sigma_t^2 J_t \Big\} dt,$$

which implies

$$dA_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{C}} \Big\{ \gamma \pi_t (\mu_t J_t + \sigma_t Z_t) + \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(J_t + U_t) - \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t \Big\} dt,$$

and (J, Z, U) is solution of BSDE (1.3.3).

Step 2: Using similar arguments as in the proof of Theorem 1.4.1, one can derive that (J, Z, U) is the *maximal solution* in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (1.3.3).

Chapter 2

Portfolio Optimization in a Default Model under Full/Partial Information

Joint paper with Marie-Claire Quenez.

Abstract: In this paper, we consider a financial market with assets exposed to some risks inducing jumps in the asset prices, and which can still be traded after default times. We use a default-intensity modeling approach, and address in this incomplete market context the problem of maximization of expected utility from terminal wealth for logarithmic, power and exponential utility functions. We study this problem as a stochastic control problem both under full and partial information. Our contribution consists in showing that the optimal strategy can be obtained by a direct approach for the logarithmic utility function, and the value function can be determined as the minimal solution of a backward stochastic differential equation for the power utility function. For the partial information case, we show how the problem can be divided into two problems: a filtering problem and an optimization problem. We also study the indifference pricing approach to evaluate the price of a contingent claim in an incomplete market and the information price for an agent with insider information.

Keywords: Optimal investment, default time, filtering, dynamic programming principle, backward stochastic differential equation, indifference pricing, information price, logarithmic utility, power utility, exponential utility.

2.1 Introduction

One of the important problems in mathematical finance is the portfolio optimization problem when the investor wants to maximize the expected utility from his terminal wealth. In this paper, we study this problem by considering a small investor on an incomplete financial market who can trade in a finite time interval $[0, T]$ by investing in risky stocks and a riskless bond. We assume that there exist some default times on the market, and each default time generates a jump of stock prices. The underlying traded assets are assumed to be some local martingales driven by a Brownian motion and a default indicating process. In such a context, we solve the portfolio optimization problem when the investor wants to maximize the expected utility from his terminal wealth. We assume that in the market there are two kinds of agents: the insider agents (the agents with insider information) and the classical agents (they only observe the asset prices and the default times). These situations are referred as full information and partial information. We will be interested not only in describing the investor's optimal utility, but also the strategies which he may follow to reach this goal.

The utility maximization problem with full information has been largely studied in the literature. In the framework of a continuous-time model the problem was studied for the first time by Merton [98]. Using the methods of stochastic optimal control, the author derives a nonlinear partial equation for the value function of the optimization problem. Some papers study this problem by using the dual problem, we can quote, for instance, Karatzas, Lehoczky and Shreve [77] for the case of complete financial models, and Karatzas *et al.* [78] and Kramkov and Schachermayer [84] for the case of incomplete financial models, they find the solution of the original problem by convex duality. These papers are useful to prove the existence of an optimal strategy in the general case, but in practice it is difficult to find the optimal strategy with the dual method. Some others study the problem by using the dynamic programming principle, we can quote Jeanblanc and Pontier [71] for a complete model with discontinuous prices, Bellamy [9] in the case of a filtration generated by a Brownian motion and a Poisson measure, Hu, Imkeller and Muller [67] for an incomplete model in the case of a Brownian filtration, and Jiao and Pham [76] in the case with a default, in which the authors study the case before the default and the case after the default.

Models with partial observation are essentially studied in the literature in a complete market framework. Detemple [47], Dothan and Feldman [48], Gennotte [63] use dynamic programming methods in a linear gaussian filtering. Lakner [86, 87] solves the optimization problem via a martingale approach and works out the special case of linear gaussian model. We mention that Frey and Runggaldier [61] and Lasry and Lions [88] study hedging problems in finance under restricted information. Pham and Quenez [110] treat the case of an incomplete stochastic volatility model. Callegaro, Di Masi and Runggaldier [32] and Roland [116] study the case of a market model with jumps.

We first study the case of full information. For the logarithmic utility function, we use a direct approach, which allows to give an expression of the optimal strategy depending uniquely on the coefficients of the model satisfied by the stocks. For the power utility function, we look for a necessary condition characterizing the value function which is solution of the maximization problem. We show that this value function is the minimal solution of a BSDE. We also give an approximation of the value function by a sequence of solutions of BSDEs. These solutions are the value functions of the maximization problem restricted to some bounded subsets of strategies. For the exponential utility function, we refer to the companion paper Lim and Quenez [91].

In order to solve the partial information problem, the common way is to use the filtering theory, so as to reduce the stochastic control problem with partial information to one with full information as in Pham and Quenez [110] or Roland [116]. Then, we can apply the results of the full information problem.

The outline of this paper is organized as follows. In Section 2, we describe the model and formulate the optimization problem. In Section 3, we solve the maximization problem for the logarithmic utility function with a direct approach. In Section 4, we consider the power utility function by giving a characterization of the value function by a BSDE thanks to the dynamic programming principle, then we approximate the value function by a sequence of solutions of Lipschitz BSDEs. In Section 5, we use results from filtering theory to reduce the stochastic control problem with partial information to one with full information, then we apply the results of the full information problem to the partial information problem. Finally, we study the indifference pricing for a contingent claim and the information price linked to the insider information.

In all this paper, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript $'$ stands for the transposition, $||.||$ the square norm, $\mathbb{1}$ the vector of $\mathbb{R}^n$ such that each component of this vector is equal to 1. Let U and V two vectors of $\mathbb{R}^n$, $U * V$ denotes the vector such that $(U*V)_i = U_iV_i$ for each $i \in \{1, \ldots, n\}$. Given a vector $X \in \mathbb{R}^n$, $|X|^2$ denotes the vector of $\mathbb{R}^n$ such that $|X|_i^2 = |X_i|^2$ for each $i \in \{1, \ldots, n\}$. For a function $f : \mathbb{R} \to \mathbb{R}$ and a vector $X \in \mathbb{R}^n$, we denote by $f(X)$ the vector of $\mathbb{R}^n$ such that $f(X)_i = f(X_i)$ for each $i \in \{1, \ldots, n\}$. Let $X \in \mathbb{R}^n$, $diag(X)$ is the matrix such that $diag(X)_{ij} = X_i$ if $i = j$ else $diag(X)_{ij} = 0$. Given a matrix M of $\mathbb{R}^{n\times p}$ we denote by $M^{\cdot j}$ the vector of $\mathbb{R}^n$ such that $M_i^{\cdot j} = M_{ij}$ for each $i \in \{1, \ldots, n\}$.

2.2 The model

We start with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a time horizon $T \in (0, \infty)$. We assume throughout that all processes are defined on the finite time interval $[0, T]$. Suppose that this space is equipped with two stochastic processes: an n-dimensional Brownian motion (W_t) and a p-dimensional jump process $(N_t) = ((N_t^i), 1 \leq i \leq p)$ with $N_t^i = \mathbb{1}_{\tau_i \leq t}$, where $(\tau_i)_{1 \leq i \leq p}$ are p default times. We make the following assumptions on the default times:

Assumption 2.2.1. (i) The defaults do not appear simultaneously: $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$.

(ii) Each default can appear at any time: $\mathbb{P}(\tau_i > t) > 0$.

We denote by $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ the filtration generated by these processes, which is assumed to satisfy the usual conditions of right-continuity and completeness. We denote for each $i \in \{1, \dots, p\}$ by (M_t^i) the compensated martingale of the process (N_t^i) and by (Λ_t^i) its compensator in the filtration $\mathbb{F}$. We assume that the compensator (Λ_t^i) is absolutely continuous with respect to the Lebesgue measure, so that there exists a process (λ_t^i) such that $\Lambda_t^i = \int_0^t \lambda_s^i ds$. We can see that for each $i \in \{1, \dots, p\}$

$$M_t^i = N_t^i - \int_0^t \lambda_s^i ds \tag{2.2.1}$$

is an $\mathbb{F}$-martingale. We assume that the process (λ_t^i) is uniformly bounded. It should be noted that the construction of such process (N_t^i) is fairly standard; see, for example, Bielecki and Rutkowski [16].

We introduce some sets used throughout the paper

- $L^{1,+}$ is the set of positive $\mathbb{F}$-adapted càd-làg processes on $[0, T]$ such that $\mathbb{E}[Y_t] < \infty$ for any $t \in [0, T]$.
- $\mathcal{S}^2$ is the set of $\mathbb{F}$-adapted càd-làg processes on $[0, T]$ such that $\mathbb{E}[\sup_{t \in [0,T]} |Y_t|^2] < \infty$.
- $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{F}$-predictable processes on $[0, T]$ such that
$$\mathbb{E}\Big[\int_0^T ||Z_t||^2 dt\Big] < \infty \ \ \text{(resp. } \int_0^T ||Z_t||^2 dt < \infty, \ \mathbb{P} - a.s. \text{)}.$$
- $L^2(M)$ (resp. $L^1_{loc}(M)$) is the set of $\mathbb{F}$-predictable processes on $[0, T]$ such that
$$\mathbb{E}\Big[\int_0^T \lambda_t' |U_t|^2 dt\Big] < \infty \ \ \text{(resp. } \int_0^T \lambda_t' |U_t| dt < \infty, \ \mathbb{P} - a.s. \text{)}.$$

We consider a financial market consisting of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each date, and n risky assets with n-dimensional price process $S = (S^1, \dots, S^n)'$ evolving according to the following model

$$dS_t = \text{diag}(S_{t^-})(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \quad 0 \leq t \leq T. \tag{2.2.2}$$

We shall make the following standing assumptions:

Assumption 2.2.2. – μ (resp. σ, β) is a $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times p}$)-valued uniformly bounded predictable stochastic process.

– For all $t \in [0,T]$, the $n \times n$ matrix σ_t is nonsingular, and we assume that $\sigma\sigma'$ is uniformly elliptic, i.e. $\epsilon I_n \leq \sigma\sigma' \leq K I_n$, $\mathbb{P} - a.s.$, for constants $0 < \epsilon < K$.

– We suppose that the process (S_t) is positive $\forall\, t \in [0,T]$, $\mathbb{P} - a.s.$

Remark 2.2.1. The assumption $\sigma\sigma'$ is uniformly elliptic implies that the predictable $\mathbb{R}^n$-valued process $\rho_t = \sigma_t^{-1}\mu_t$ is uniformly bounded.

An n-dimensional $\mathbb{F}$-predictable process $\pi = (\pi_t)_{0\leq t\leq T}$ is called trading strategy if $\int \frac{\pi_t^i X_t}{S_{t-}^i} dS_t^i$ is well defined for each $i \in \{1,\dots,n\}$. For $i \in \{1,\dots,n\}$, the process π_t^i describes the part of the wealth invested in asset i at time t. The number of shares of asset i is given by $\frac{\pi_t^i X_t}{S_{t-}^i}$. The wealth process $X^{x,\pi}$ associated with a trading strategy π and an initial capital x, under the assumption that the trading strategy is self-financing, satisfies

$$X_t^{x,\pi} = x \exp\Big(\int_0^t \big(\pi_s'\mu_s - \frac{||\pi_s'\sigma_s||^2}{2}\big)ds + \int_0^t \pi_s'\sigma_s dW_s\Big)\prod_{j=1}^p (1 + \pi_{\tau^j}'\beta_{\tau^j}^{.,j} N_t^j). \tag{2.2.3}$$

For a given initial time t and an initial capital x, the associated wealth process is denoted by $X_s^{t,x,\pi}$.

Now let $U : \mathbb{R} \to \mathbb{R}$ be a utility function. The optimization problem consists in maximizing the expected utility from terminal wealth over the class $\mathcal{A}(x)$ of admissible portfolios (which will be defined in the sequel). More precisely, we want to characterize the value function of this problem, which is defined by

$$V(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\big[U(X_T^{x,\pi})\big], \tag{2.2.4}$$

and we also want to give the optimal strategy when it exists. We begin by the simple case when U is the logarithmic utility function, then we study the case of power utility function.

2.3 Logarithmic utility function

In this section, we specify the meaning of optimality for trading strategies by stipulating that the agent wants to maximize his expected utility from his terminal wealth $X_T^{x,\pi}$ with respect to the logarithmic utility function

$$U(x) = \log(x),\ x > 0.$$

Our goal is to solve the following optimization problem (we take $n = p = 1$ for the sake of simplicity)

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\big[\log(X_T^{x,\pi})\big], \tag{2.3.1}$$

with $\mathcal{A}(x)$ the set of admissible portfolios defined by:

Definition 2.3.1. The set of admissible trading strategies $\mathcal{A}(x)$ consists of all $\mathbb{F}$-predictable processes $(\pi_t)_{0\le t\le T}$ satisfying $\mathbb{E}[\int_0^T |\pi_t\sigma_t|^2 dt] + \mathbb{E}[\int_0^T \lambda_t|\log(1+\pi_t\beta_t)|dt] < \infty$, and such that $\pi_t\beta_t > -1,\ \mathbb{P}-a.s.$, for any $0 \le t \le \tau$.

We can see from (2.3.1) that $V(x) = \log(x) + V(1)$. Hence, we only study the case $x = 1$. And for the sake of brevity, we shall denote X_t^π instead of $X_t^{1,\pi}$ and $\mathcal{A}$ instead of $\mathcal{A}(1)$.

Remark 2.3.1. The condition $\pi_t\beta_t > -1,\ \mathbb{P}-a.s.$, for any $0 \le t \le \tau$ is stronger than $X_t^{x,\pi} > 0,\ \mathbb{P}-a.s.$, for any $0 \le t \le T$, but it is necessary to be able to write $\log(X_t^\pi)$ under the form

$$\log(X_t^\pi) = \int_0^t \Big(\pi_s\mu_s - \frac{|\pi_s\sigma_s|^2}{2}\Big)ds + \int_0^t \pi_s\sigma_s dW_s + \int_0^t \log(1+\pi_s\beta_s)(dM_s + \lambda_s ds), \tag{2.3.2}$$

this form is useful to solve the maximization problem with a direct approach.

As in Kramkov and Schachermayer [84], we assume that $\sup_{\pi\in\mathcal{A}} \mathbb{E}[\log(X_T^\pi)] < \infty$.

We add the following assumption on the coefficients to be able to solve the optimization problem (2.3.1) directly:

Assumption 2.3.1. The process (β_t^{-1}) is uniformly bounded.

With this assumption, we get easily the value function $V(x)$ and the optimal strategy:

Theorem 2.3.1. *The solution of the optimization problem (2.3.1) is given by*

$$V(x) = \log(x) + \mathbb{E}\Big[\int_0^T \Big(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t\log(1+\hat{\pi}_t\beta_t)\Big)dt\Big],$$

with $\hat{\pi}$ the optimal trading strategy defined by

$$\hat{\pi}_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \textit{if } t < \tau \textit{ and } \beta_t \neq 0, \\ \dfrac{\mu_t}{\sigma_t^2} \quad \textit{if } t < \tau \textit{ and } \beta_t = 0 \textit{ or } t \ge \tau. \end{cases} \tag{2.3.3}$$

Proof. With (2.3.2) and Definition 2.3.1, we get the following expression for $V(1)$

$$V(1) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\Big[\int_0^T \Big(\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t \log(1+\pi_t\beta_t)\Big)dt\Big],$$

which implies that

$$V(1) \le \mathbb{E}\Big[\int_0^T \operatorname*{ess\,sup}_{\pi_t\beta_t > -1} \Big\{\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t \log(1+\pi_t\beta_t)\Big\}dt\Big]. \tag{2.3.4}$$

For any $t \in [0,T]$ and any $\omega \in \Omega$, we have

$$\operatorname*{ess\,sup}_{\pi_t\beta_t > -1} \Big\{\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t \log(1+\pi_t\beta_t)dt\Big\} = \hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t),$$

with $\hat{\pi}_t$ defined by (2.3.3). Then, from inequality (2.3.4), we can see that

$$V(1) \le \mathbb{E}\Big[\int_0^T \Big(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t)\Big)dt\Big].$$

It now is sufficient to show that the strategy $(\hat{\pi}_t)$ is admissible. It is clearly the case with Assumption 2.3.1. Thus, the previous inequality is an equality

$$V(1) = \mathbb{E}\Big[\int_0^T \Big(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t \log(1+\hat{\pi}_t\beta_t)\Big)dt\Big],$$

and the strategy $(\hat{\pi}_t)$ is optimal. □

Remark 2.3.2. Assumption 2.3.1 can be reduced to

$$\mathbb{E}\Big[\int_0^T |\hat{\pi}_t\sigma_t|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \lambda_t |\log(1+\hat{\pi}_t\beta_t)| dt\Big] < \infty.$$

Remark 2.3.3. Recall that in the case without default, the optimal strategy is given by $\pi_t^0 = \mu_t/\sigma_t^2$. Thus, in the case of default, the optimal strategy can be written under the form

$$\hat{\pi}_t = \pi_t^0 - \epsilon_t,$$

where ϵ_t is an additional term given by

$$\epsilon_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} + \dfrac{1}{2\beta_t} - \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{if } t < \tau \text{ and } \beta_t \neq 0, \\ 0 \quad \text{if } t < \tau \text{ and } \beta_t = 0 \text{ or } t \ge \tau. \end{cases}$$

Note that if we assume that β_t is negative (resp. β_t is positive), i.e. the asset price (S_t) has a negative jump (resp. a positive jump) at default time τ, ϵ_t is positive (resp. negative), i.e. the agent has to invest less (resp. more) in the risky asset than in the case of a market without default.

2.4 Power utility

In this section, we keep the notation of Section 2.3, and we shall study the case of the power utility function defined by

$$U(x) = x^\gamma, \quad x \geq 0, \quad \gamma \in (0,1),$$

where γ is a given constant, which can be seen as a coefficient of absolute risk aversion.

In order to formulate the optimization problem, we first define the set of admissible trading strategies.

Definition 2.4.1. The set of admissible trading strategies $\mathcal{A}(x)$ consists of all $\mathbb{F}$-predictable processes $\pi = (\pi_t)_{0 \leq t \leq T}$ such that $\int_0^T ||\pi_t' \sigma_t||^2 dt + \int_0^T |\pi_t' \beta_t| \lambda_t dt < \infty$, $\mathbb{P} - a.s.$, and such that $\pi_{\tau_j}' \beta_{\tau_j}^{:,j} \geq -1$, $\mathbb{P} - a.s.$, for each $j \in \{1, \ldots, p\}$.

Remark 2.4.1. From expression (2.2.3), it is obvious that the condition $\pi_{\tau_j}' \beta_{\tau_j}^{:,j} \geq -1$, $\mathbb{P} - a.s.$, for each $j \in \{1, \ldots, p\}$ is equivalent to $X_t^{x,\pi} \geq 0$, $\mathbb{P} - a.s.$, for any $t \in [0,T]$.

The portfolio optimization problem consists in determining a predictable portfolio $\pi_t = (\pi_t^1, \ldots, \pi_t^n)'$ which attains the optimal value

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\big[\big(X_T^{x,\pi}\big)^\gamma\big]. \tag{2.4.1}$$

Problem (2.4.1) can be clearly written as $V(x) = x^\gamma V(1)$. Therefore, it is sufficient to study the case $x = 1$. As in [84], we assume that $\sup_{\pi \in \mathcal{A}(1)} \mathbb{E}[(X_T^{1,\pi})^\gamma] < \infty$.

To solve the optimization problem, we give a dynamic extension of the initial problem. For any initial time $t \in [0,T]$, we define the value function $J(t)$ by the following random variable

$$J(t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t(1)} \mathbb{E}\big[\big(X_T^{t,1,\pi}\big)^\gamma \big| \mathcal{F}_t\big],$$

with $\mathcal{A}_t(1)$ the set of $\mathbb{F}$-predictable processes $\pi = (\pi_s)_{t \leq s \leq T}$ such that $\int_t^T ||\pi_s' \sigma_s||^2 ds + \int_t^T |\pi_s' \beta_s| \lambda_s ds < \infty$, $\mathbb{P} - a.s.$, and such that $\pi_{\tau_j}' \beta_{\tau_j}^{:,j} \geq -1$, $\mathbb{P} - a.s.$, for each $j \in \{1, \ldots, p\}$.

For the sake of brevity, we shall denote X_s^π (resp. $X_s^{t,\pi}$) instead of $X_s^{1,\pi}$ (resp. $X_s^{t,1,\pi}$) and $\mathcal{A}$ (resp. $\mathcal{A}_t$) instead of $\mathcal{A}(1)$ (resp. $\mathcal{A}_t(1)$). And to simplify the notation, we suppose in the sequel of this section that $n = p = 1$, we give the generalization of the results in Part 2.4.3.

In the sequel, we will use the martingale representation theorem (see for example Jeanblanc *et al.* [73]) to characterize the value function $J(t)$:

Lemma 2.4.1. *Any* $(\mathbb{P}, \mathbb{F})$*-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s dW_s + \int_0^t b_s dM_s, \ \forall\, t \in [0,T], \ \mathbb{P} - a.s., \tag{2.4.2}$$

where $a \in L^2_{loc}(W)$ and $b \in L^1_{loc}(M)$. If (m_t) is a square integrable martingale, each term on the right-hand side of the representation (2.4.2) is square integrable.

2.4.1 Optimization over bounded strategies

Before studying the value function $J(t)$, we study the value functions $(J^k(t))_{k\in\mathbb{N}}$ defined by

$$J^k(t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k_t} \mathbb{E}\big[\big(X_T^{t,\pi}\big)^\gamma\big|\mathcal{F}_t\big], \ \forall\, t \in [0,T], \ \mathbb{P}-a.s., \tag{2.4.3}$$

where $\mathcal{A}^k_t$ is the set of strategies of $\mathcal{A}_t$ uniformly bounded by k. This means that the part of the wealth invested in the asset has to be bounded by a constant k (which makes sense in finance, because the ratio of the amount of money invested or borrowed to the wealth must be bounded according to the financial legislation).

Let us fix $k \in \mathbb{N}$. We want to characterize the value function $J^k(t)$ by a BSDE. For that, we introduce for any $\pi \in \mathcal{A}^k$ the càd-làg process (J^π_t) defined for all $t \in [0,T]$ by

$$J^\pi_t = \mathbb{E}\big[\big(X_T^{t,\pi}\big)^\gamma\big|\mathcal{F}_t\big].$$

The family $((J^\pi_t))_{\pi\in\mathcal{A}^k}$ is uniformly bounded:

Lemma 2.4.2. *For any $\pi \in \mathcal{A}^k$, the process (J^π_t) is uniformly bounded by a constant independent of π.*

Proof. Fix $t \in [0,T]$. We have

$$J^\pi_t = \mathbb{E}\Big[\exp\Big(\gamma\int_t^T(\mu_s\pi_s - \frac{|\sigma_s\pi_s|^2}{2})ds + \int_t^T \gamma\sigma_s\pi_s dW_s\Big)(1+\pi_\tau\beta_\tau\mathbb{1}_{t<\tau\leq T})^\gamma\Big|\mathcal{F}_t\Big],$$

since the coefficients μ_t, σ_t and β_t are supposed to be bounded, we have

$$J^\pi_t \leq (1+k\,|\beta|_\infty)^\gamma \exp\Big((\gamma\, k\,|\mu|_\infty + \gamma^2\frac{(k\,|\sigma|_\infty)^2}{2})\,T\Big).$$

□

Classically, for any $\pi \in \mathcal{A}^k$ the process (J^π_t) can be shown to be the solution of a linear BSDE. More precisely, there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$, such that (J^π, Z^π, U^π) is the solution in $\mathcal{S}^2 \times L^2(W) \times L^2(M)$ of the linear BSDE with bounded coefficients

$$\begin{cases} -\,dJ^\pi_t &= -\,Z^\pi_t dW_t - U^\pi_t dM_t + \Big\{\gamma\pi_t(\mu_t J^\pi_t + \sigma_t Z^\pi_t) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J^\pi_t \\ &\qquad + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J^\pi_t + U^\pi_t)\Big\}dt, \\ J^\pi_T &= 1. \end{cases} \tag{2.4.4}$$

Using the fact that $J^k(t) = \operatorname{ess\,sup}_{\pi\in\mathcal{A}^k_t} J^\pi_t$ for any $t \in [0,T]$, we derive that $(J^k(t))$ corresponds to the solution of a BSDE, whose driver is the essential supremum over π of the drivers of $(J^\pi_t)_{\pi\in\mathcal{A}^k}$. More precisely,

Proposition 2.4.1. *The following properties hold*

- *Let* (Y, Z, U) *be the solution in* $\mathcal{S}^2 \times L^2(W) \times L^2(M)$ *of the following Lipschitz BSDE*

$$\begin{cases} -\,dY_t = -\,Z_t dW_t - U_t dM_t + \operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k} \Big\{ \gamma \pi_t(\mu_t Y_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 Y_t \\ \qquad\qquad + \lambda_t((1+\pi_t \beta_t)^\gamma - 1)(Y_t + U_t) \Big\} dt, \\ Y_T = 1. \end{cases} \tag{2.4.5}$$

  Then, $J^k(t) = Y_t,\ \mathbb{P} - a.s.$, *for any* $t \in [0, T]$.

- *There exists a unique optimal strategy for* $J^k(0) = \sup_{\pi \in \mathcal{A}^k} \mathbb{E}[(X_T^\pi)^\gamma]$.

- *A strategy* $\hat{\pi} \in \mathcal{A}^k$ *is optimal for* $J^k(0)$ *if and only if it attains the essential supremum of the driver in (2.4.5)* $dt \otimes d\mathbb{P} - a.e.$

Proof. Since for any $\pi \in \mathcal{A}^k$ there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$ such that (J^π, Z^π, U^π) is the solution of the BSDE

$$-\,dJ_t^\pi = f^\pi(t, J_t^\pi, Z_t^\pi, U_t^\pi)dt - Z_t^\pi dW_t - U_t^\pi dM_t\ ;\ J_T^\pi = 1,$$

with $f^\pi(s, y, z, u) = \frac{\gamma(\gamma-1)}{2}\pi_s^2 \sigma_s^2 y + \gamma \pi_s(\mu_s y + \sigma_s z) + \lambda_s((1 + \pi_s \beta_s)^\gamma - 1)(y + u)$. Let us introduce the driver f which satisfies $ds \otimes d\mathbb{P} - a.e.$

$$f(s, y, z, u) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k} f^\pi(s, y, z, u).$$

Note that f is Lipschitz, since the supremum of affine functions, whose coefficients are bounded by a constant $c > 0$, is Lipschitz with Lipschitz constant c. Hence, by results of Tang and Li [126], the BSDE with Lipschitz driver f

$$-\,dY_t = f(y, Y_t, Z_t, U_t)dt - Z_t dW_t - U_t dM_t\ ;\ Y_T = 1$$

admits a unique solution denoted by (Y, Z, U).
By the comparison theorem in case of jumps (see for example Royer [118]) $Y_t \geq J_t^\pi$, $\forall\, t \in [0, T]$, $\mathbb{P} - a.s.$ As this inequality is satisfied for any $\pi \in \mathcal{A}^k$, it is obvious that $Y_t \geq \operatorname{ess\,sup}_{\pi \in \mathcal{A}^k} J_t^\pi$, $\mathbb{P} - a.s.$ Also, by applying a predictable selection theorem, one can easily show that there exists $\hat{\pi} \in \mathcal{A}^k$ such that for any $t \in [0, T]$, we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k} \Big\{ \gamma \pi_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 Y_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t + U_t) \Big\} \\ &\qquad\qquad = \gamma \hat{\pi}_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \hat{\pi}_t^2 \sigma_t^2 Y_t + \lambda_t((1+\hat{\pi}_t\beta_t)^\gamma - 1)(Y_t + U_t). \end{aligned}$$

Thus (Y, Z, U) is a solution of BSDE (2.4.4) associated with $\hat{\pi}$. Therefore, by uniqueness of the solution of BSDE (2.4.4), we have $Y_t = J_t^{\hat{\pi}}$ and thus $Y_t = \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t^k} J_t^{\pi} = J_t^{\hat{\pi}}$, $\forall\, t \in [0,T]$, $\mathbb{P} - a.s.$

The uniqueness of the optimal strategy is due to the strict concavity of the function $x \mapsto x^{\gamma}$. □

2.4.2 General case

In this part, we characterize the value function $J(t)$ by a BSDE, but the general case is more complicated than the case with bounded strategies, and it needs more technical tools. Note that the random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent, and that the process $(J(t))$ is adapted but not necessarily progressive. Using dynamic control techniques, we derive the following characterization of the value function:

Proposition 2.4.2. *$(J(t))$ is the smallest $\mathbb{F}$-adapted process such that $((X_t^{\pi})^{\gamma} J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$ with the terminal condition $J(T) = 1$. More precisely, if $(\bar{J}_t)$ is an $\mathbb{F}$-adapted process such that $((X_t^{\pi})^{\gamma}(\bar{J}_t))$ is a supermartingale for any $\pi \in \mathcal{A}$ with the terminal condition $\bar{J}_T = 1$, then for any $t \in [0,T]$, we have $J(t) \leq \bar{J}_t$, $\mathbb{P} - a.s.$*

From [84], there exists an optimal strategy $\hat{\pi} \in \mathcal{A}$ such that $J(0) = \mathbb{E}[(X_T^{\hat{\pi}})^{\gamma}]$. And with the dynamic programming principle, we have the following optimality criterion:

Proposition 2.4.3. *The following assertions are equivalent:*

i) $\hat{\pi}$ is an optimal strategy, that is $\mathbb{E}[(X_T^{\hat{\pi}})^{\gamma}] = \sup_{\pi \in \mathcal{A}} \mathbb{E}[(X_T^{\pi})^{\gamma}]$.

ii) The process $((X_t^{\hat{\pi}})^{\gamma} J(t))$ is a martingale.

The proof of these propositions is given in Appendix 2.6.1.

By Proposition 2.4.2, $(J(t))$ is a supermartingale. Hence, $\mathbb{E}[J(t)] \leq J(0) < \infty$ for any $t \geq 0$.

Proposition 2.4.4. *There exists a càd-làg modification of $J(t)$ which is denoted by (J_t).*

Proof. By Proposition 2.4.3, we know that $J(t) = \mathbb{E}[(X_T^{\hat{\pi}})^{\gamma} | \mathcal{F}_t]/(X_t^{\hat{\pi}})^{\gamma}$, $\mathbb{P} - a.s.$ Which implies the desired result. □

This càd-làg process is characterized by a BSDE. More precisely,

Theorem 2.4.1. *There exist $Z \in L^2_{loc}(W)$ and $U \in L^1_{loc}(M)$ such that the process (J, Z, U) is the minimal solution*[1] *in $L^{1,+} \times L^2_{loc}(W) \times L^1_{loc}(M)$ of the following BSDE*

$$\begin{cases} -\,dJ_t &= -\,Z_t dW_t - U_t dM_t + \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{ \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J_t \\ & \qquad + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t + U_t)\Big\} dt, \\ J_T &= 1. \end{cases} \tag{2.4.6}$$

There exists a unique optimal strategy for $J_0 = \sup_{\pi\in\mathcal{A}} \mathbb{E}[(X_T^\pi)^\gamma]$. Moreover, $\hat{\pi} \in \mathcal{A}$ is optimal if and only if it attains the essential supremum of the driver in (2.4.6) $dt \otimes d\mathbb{P} - a.e.$

The proof of this theorem is postponed in Appendix 2.6.2.

There exists another characterization of the value function (J_t) as the limit of processes $(J^k(t))_{k\in\mathbb{N}}$ as k tends to $+\infty$, where $(J^k(t))$ is the value function in the case where the strategies are bounded by k:

Theorem 2.4.2. *For any $t \in [0, T]$, we have*

$$J_t = \lim_{k\to\infty} \uparrow J^k(t), \ \mathbb{P} - a.s.$$

The proof of this theorem is given in Appendix 2.6.3.

This allows to approximate the value function J_t by numerical computations, since the value functions $(J^k(t))_{k\in\mathbb{N}}$ are the solution of Lipschitz BSDEs and the results of Bouchard and Elie [22] can be applied.

2.4.3 Several default times and several assets

In this part, we only give the BSDEs in the case of several default times and several assets. The proofs are not given, but they are identical to the proofs for $n = p = 1$.

– BSDE (2.4.5) is written

$$\begin{cases} -\,dY_t &= -\,Z_t' dW_t - U_t' dM_t + \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k} \Big\{ \gamma\pi_t'(Y_t\mu_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}||\pi_t'\sigma_t||^2 Y_t \\ & \qquad + [(\mathbb{1} + \pi_t'\beta_t)^\gamma - \mathbb{1}](Y_t\lambda_t + \lambda_t * U_t))\Big\} dt, \\ Y_T &= 1, \end{cases}$$

[1]That is for any solution $(\bar{J}, \bar{Z}, \bar{U})$ of BSDE (2.4.6) in $L^{1,+} \times L^2_{loc}(W) \times L^1_{loc}(M)$, we have $J_t \le \bar{J}_t, \ \forall\, t \in [0, T], \ \mathbb{P} - a.s.$

– and BSDE (2.4.6) is written

$$\begin{cases} -dY_t &= -Z_t'dW_t - U_t'dM_t + \operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\Big\{\gamma\pi_t'(Y_t\mu_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}||\pi_t'\sigma_t||^2 Y_t \\ & \quad + [(\mathbb{1} + \pi_t'\beta_t)^\gamma - \mathbb{1}](Y_t\lambda_t + \lambda_t * U_t))\Big\}dt, \\ Y_T &= 1. \end{cases}$$

2.5 The partial information case

The difference between this section and the previous sections is that here we require the investment process to be adapted to the natural filtration generated by the price process and the default times process. This requirement means that the only available information for agents in this economy at a certain time are the price of the financial assets up to that time and the default times. The underlying Brownian motion, the drift process and the compensator process in the system of equations for the asset prices are not directly observable.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability triplet and $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ a filtration in $\mathcal{F}$ satisfying the usual conditions (augmented and right continuous). Suppose that this space is equipped with (W_t) and (N_t) as in Section 2.2. We also assume there are one risk-free asset and n risky assets on the market. As in Section 2.2, we assume that the price process $(S_t)_{0\le t\le T}$ evolves according to the following model

$$dS_t = \operatorname{diag}(S_{t^-})(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \quad 0 \le t \le T, \tag{2.5.1}$$

moreover, we assume that $\sigma_t = \sigma(t, S_{t^-}, t\wedge\tau)$ and $\beta_t = \beta(t, S_{t^-}, t\wedge\tau)$, with $t\wedge\tau = (t\wedge\tau_1, \dots, t\wedge\tau_p)'$. The known functions $\sigma(t,s,h)$ and $\beta(t,s,h)$ are measurable mappings from $[0,T]\times\mathbb{R}^n\times\mathbb{R}^p$ into $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times p}$. We make the hypotheses of Assumption 2.2.2 and we add the following assumption:

Assumption 2.5.1. The functions $s\sigma(t,s,h)$ and $s\beta(t,s,h)$ are Lipschitz in $s\in\mathbb{R}^n$, uniformly in $t\in[0,T]$ and $h\in\mathbb{R}^p$.

We now consider an agent in this market who can observe neither the Brownian motion (W_t) nor the drift (μ_t) and the process (λ_t), but only the asset price process (S_t) and the default times $(\tau_i)_{1\le i\le p}$. We shall denote by $\mathbb{G} = \{\mathcal{G}_t, 0\le t\le T\}$ the $\mathbb{P}$-filtration augmented by the price process (S_t) and the default process (N_t). The trading strategies are defined as in Section 2.2, but we add the condition that they are $\mathbb{G}$-predictable. We now want to solve the problem of maximization of expected utility from terminal wealth for logarithmic, power and exponential utility functions. It is not possible to use directly the results of the full information case because we do not know the Brownian motion, the drift and the compensator. Moreover, there exists no martingale representation theorem for the $\mathbb{G}$-martingales. Thus, before to study the problem of maximization, we begin by an operation of filtering as in Pham and Quenez [110].

2.5.1 Filtering

Consider the positive local martingale defined by $L_0 = 1$ and $dL_t = -L_t\,\rho_t' dW_t$. It is explicitly given by

$$L_t = \exp\Big(- \int_0^t \rho_s' dW_s - \frac{1}{2}\int_0^t ||\rho_s||^2 ds\Big). \tag{2.5.2}$$

Since with Assumption 2.2.2 the process (ρ_t) is uniformly bounded, we have:

Lemma 2.5.1. *The process (L_t) is a martingale.*

Therefore, one can define a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ characterized by

$$\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\Big|_{\mathcal{F}_t} = L_t,\ \ 0 \le t \le T. \tag{2.5.3}$$

By Girsanov's theorem, the n-dimensional process defined by

$$\tilde{W}_t = W_t + \int_0^t \rho_s ds \tag{2.5.4}$$

is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion and the compensated martingale (M_t) is still a $(\mathbb{Q}, \mathbb{F})$-martingale. The dynamics of (S_t) under $\mathbb{Q}$ is given by

$$dS_t = \mathrm{diag}(S_{t^-})(\sigma(t, S_{t^-}, t\wedge\tau)d\tilde{W}_t + \beta(t, S_{t^-}, t\wedge\tau)dN_t). \tag{2.5.5}$$

We begin by proving a proposition which will be of paramount importance in the sequel:

Proposition 2.5.1. *Under Assumption 2.2.2 and with Lemma 2.5.1, the filtration $\mathbb{G}$ is the augmented filtration of $(\tilde{W}, N)$.*

Proof. Let $\mathbb{F}^{\tilde{W},N}$ be the augmented filtration of $(\tilde{W}, N)$. From (2.5.5), we have

$$\tilde{W}_t = \int_0^t \sigma_s^{-1}\mathrm{diag}(S_t^{-1})dS_s - \int_0^t \sigma_s^{-1}\beta_s dN_s,$$

for all $t \in [0, T]$, which implies that $(\tilde{W}_t)$ is $\mathbb{G}$-adapted and $\mathbb{F}^{\tilde{W},N} \subset \mathbb{G}$. Conversely, under the assumptions on the coefficients, by a classical result of stochastic differential equation (see Protter [115], Theorem V 3.7), the unique solution of (2.5.5) is $\mathbb{F}^{\tilde{W},N}$-adapted, hence $\mathbb{G} \subset \mathbb{F}^{\tilde{W},N}$ and finally $\mathbb{G} = \mathbb{F}^{\tilde{W},N}$. □

Since the processes (ρ_t) and (λ_t) are not $\mathbb{G}$-predictable, it is natural to introduce the $\mathbb{G}$-conditional law of the random variables ρ_t and λ_t, say

$$\tilde{\lambda}_t = \mathbb{E}\big[\lambda_t \big| \mathcal{G}_t\big] \text{ and } \tilde{\rho}_t = \mathbb{E}\big[\rho_t \big| \mathcal{G}_t\big].$$

Consider the couple of processes $(\bar{W}_t, \bar{M}_t)$ defined by

$$\begin{cases} \bar{W}_t = \tilde{W}_t - \int_0^t \tilde{\rho}_s ds, \\ \bar{M}_t = N_t - \int_0^t \tilde{\lambda}_s ds. \end{cases} \tag{2.5.6}$$

These are the so-called innovation processes of filtering theory. By classical results in filtering theory (see for example Pardoux [102], Proposition 2.27), we have:

Proposition 2.5.2. *The process $(\bar{M}_t)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.*

Proof. Since the process (N_t) and the intensity $(\tilde{\lambda}_t)$ are $\mathbb{G}$-adapted, the process $(\bar{M}_t)$ is $\mathbb{G}$-adapted. We can write from (2.2.1)

$$\bar{M}_t = M_t + \int_0^t (\lambda_s - \tilde{\lambda}_s) ds.$$

By the law of iterated conditional expectation, it is easy to check that $(\bar{M}_t)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale. □

Remark 2.5.1. From Proposition 2.5.1 and (2.5.6), the filtration $\mathbb{G}$ is equal to the augmented filtration of $(\tilde{W}, \bar{M})$, since $[\bar{M}]_t = N_t$.

We have also the following property about the process $(\bar{W}_t)$:

Proposition 2.5.3. *The process $(\bar{W}_t)$ is a $(\mathbb{P}, \mathbb{G})$-Brownian motion.*

Proof. We can write with (2.5.4)

$$\bar{W}_t = W_t + \int_0^t \sigma_s^{-1} (\mu_s - \tilde{\mu}_s) ds, \tag{2.5.7}$$

where $\tilde{\mu}_t = \mathbb{E}\big[\mu_t \big| \mathcal{G}_t\big]$. By Proposition 2.5.1, $\bar{W}$ is $\mathbb{G}$-adapted. Moreover, we have $[\bar{W}^i, \bar{W}^j]_t = \delta_{ij} t$ for all $t \in [0, T]$, where δ_{ij} is the Kronecker notation. By the law of iterated conditional expectation, it is easy to check that $(\bar{W}_t)$ is a $\mathbb{G}$-martingale. We then conclude by Levy's characterization theorem on Brownian motion (see, e.g., Theorem 3.3.16 in Karatzas and Shreve [79]). □

Denote by (Λ_t), the $(\mathbb{Q}, \mathbb{F})$-martingale given by $\Lambda_t = 1/L_t$. We then have

$$\frac{d\,\mathbb{P}}{d\,\mathbb{Q}}\Big|_{\mathcal{F}_t} = \Lambda_t, \ 0 \leq t \leq T.$$

Let $(\tilde{\Lambda}_t)$ be the $(\mathbb{Q}, \mathbb{G})$-martingale given by $\tilde{\Lambda}_t = \mathbb{E}_{\mathbb{Q}}\big[\Lambda_t \big| \mathcal{G}_t\big]$. Recall the classical proposition (see for example Lakner [87] or [110]), which gives the expression of $(\tilde{\Lambda}_t)$:

Lemma 2.5.2. *Since with Assumption 2.2.2 the process (ρ_t) is uniformly bounded, we have*

$$\tilde{\Lambda}_t = \exp\Big(\int_0^t \tilde{\rho}_s' d\tilde{W}_s - \frac{1}{2}\int_0^t ||\tilde{\rho}_s||^2 ds\Big). \tag{2.5.8}$$

Proposition 2.5.4. *The process $(\bar{M}_t)$ is a $(\mathbb{P}, \mathbb{G})$-martingale.*

Proof. Since $\frac{d\,\mathbb{P}}{d\,\mathbb{Q}}\big|_{\mathcal{G}_t} = \tilde{\Lambda}_t$, we can apply Girsanov's theorem and we get that the process $(\bar{M}_t)$ is a $(\mathbb{P}, \mathbb{G})$-martingale. □

By means of innovation processes, we can describe from (2.5.1) and (2.5.7) the dynamics of the partially observed model within a framework of full observation model

$$\begin{cases} dS_t = diag(S_{t^-})(\tilde{\mu}_t dt + \sigma(t, S_{t^-}, t \wedge \tau) d\bar{W}_t + \beta(t, S_{t^-}, t \wedge \tau) dN_t), \\ d\bar{M}_t = dN_t - \tilde{\lambda}_t dt, \end{cases} \tag{2.5.9}$$

where $(\tilde{\mu}_t)$ and $(\tilde{\lambda}_t)$ are $\mathbb{G}$-predictable processes.
Hence, the operations of filtering and control can be put in sequence and thus separated.

2.5.2 Optimization problem for the logarithmic and power utility functions

To apply the results of Section 2.4, it is sufficient to have a martingale representation theorem for $(\mathbb{P}, \mathbb{G})$-martingales with respect to $\bar{W}$ and $\bar{M}$. Notice it cannot be directly derived from the usual martingale representation theorem since $\mathbb{G}$ is not equal to the filtration generated by $\bar{W}$ and $\bar{M}$.

Lemma 2.5.3. *Any $(\mathbb{P}, \mathbb{G})$-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s' d\bar{W}_s + \int_0^t b_s' d\bar{M}_s, \ \forall\, t \in [0, T], \ \mathbb{P} - a.s., \tag{2.5.10}$$

where $a \in L^2_{loc}(\bar{W})$ and $b \in L^1_{loc}(\bar{M})$. If $(m_t)_{0 \leq t \leq T}$ is a square integrable martingale, each term on the right-hand side of the representation (2.5.10) is square integrable.

The proof of this lemma is postponed in Appendix 2.6.4.

It is now possible to apply the previous results because the price process evolves according to the equation

$$\begin{cases} dS_t = \operatorname{diag}(S_{t^-})(\tilde{\mu}_t dt + \sigma(t, S_{t^-}, t \wedge \tau) d\bar{W}_t + \beta(t, S_{t^-}, t \wedge \tau) dN_t), \\ d\bar{M}_t = dN_t - \tilde{\lambda}_t dt, \end{cases}$$

where each coefficient is $\mathbb{G}$-predictable, and there exists a martingale representation theorem for $(\mathbb{P}, \mathbb{G})$-martingales. We get the following characterization for the value functions and the optimal strategies when they exist.

For the logarithmic utility function, we assume that the process $(\beta^{-1}(t, S_{t^-}, t \wedge \tau))$ is uniformly bounded, and we have:

Theorem 2.5.1. *The solution of the optimization problem for the logarithmic utility function is given by*

$$V(x) = \log(x) + \mathbb{E}\Big[\int_0^T \Big(\hat{\pi}_t \tilde{\mu}_t - \frac{|\hat{\pi}_t \sigma_t|^2}{2} + \tilde{\lambda}_t \log(1 + \hat{\pi}_t \beta_t) \Big) dt \Big],$$

with $\hat{\pi}$ the optimal trading strategy defined by

$$\hat{\pi}_t = \begin{cases} \dfrac{\tilde{\mu}_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\tilde{\mu}_t \beta_t + \sigma_t^2)^2 + 4\tilde{\lambda}_t \beta_t^2 \sigma_t^2}}{2\beta_t \sigma_t^2} & \text{if } t < \tau \text{ and } \beta_t \neq 0, \\ \dfrac{\tilde{\mu}_t}{\sigma_t^2} \quad \text{if } t < \tau \text{ and } \beta_t = 0 \text{ or } t \geq \tau. \end{cases}$$

Therefore, we can see that the optimal portfolio in the case of partial information can be formally derived from the full information case by replacing the unobservable coefficients μ_t and λ_t by theirs estimates $\tilde{\mu}_t$ and $\tilde{\lambda}_t$.

For the power utility function, we have:

Theorem 2.5.2.
- *Let $(\bar{Y}, \bar{Z}, \bar{U})$ be the minimal solution in $L^{1,+} \times L^2_{loc}(\bar{W}) \times L^1_{loc}(\bar{M})$ of BSDE (2.4.6) with $(W,\, M,\, \mu,\, \lambda)$ replaced by $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$, then*

$$\bar{Y}_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t} \mathbb{E}\big[(X_T^{t,\pi})^\gamma \big| \mathcal{G}_t \big], \ \mathbb{P} - a.s.$$

- *If a strategy $\hat{\pi} \in \mathcal{A}$ is optimal for $J_0 = \sup_{\pi \in \mathcal{A}} \mathbb{E}[(X_T^\pi)^\gamma]$, then $\hat{\pi}$ attains the essential supremum in the driver of BSDE (2.4.6) $dt \otimes d\mathbb{P}$ a.s. with μ_t and λ_t replaced by $\tilde{\mu}_t$ and $\tilde{\lambda}_t$.*

– *Moreover the process* $(\bar{Y}_t)$ *is the nondecreasing limit of the sequence of processes* $((\bar{Y}_t^k))_{k\in\mathbb{N}}$*, where for each* $k \in \mathbb{N}$*,* $(\bar{Y}^k, \bar{Z}^k, \bar{U}^k)$ *is the solution in* $\mathcal{S}^2 \times L^2(\bar{W}) \times L^2(\bar{M})$ *of BSDE (2.4.5) with* $(W,\ M,\ \mu,\ \lambda)$ *replaced by* $(\bar{W},\ \bar{M},\ \tilde{\mu},\ \tilde{\lambda})$*.*

2.5.3 Optimization problem for the exponential utility function and indifference pricing

We can also apply the results of Lim and Quenez [91] for the exponential utility function. In this case, we assume that the agent faces some liability, which is modeled by a random variable ξ (for example, ξ may be a contingent claim written on some default events affecting the price of the underlying assets). We suppose that ξ is a non-negative $\mathcal{G}_T$-adapted process (note that all the results still hold under the assumption that ξ is only lower bounded). Without loss of generality, we can use a somewhat different notion of trading strategy: ϕ_t corresponds to the amount of money invested in the assets. The number of shares i at time t is equal to ϕ_t^i/S_{t-}^i. With this notation, under the assumption that the trading strategy is self-financing, the wealth process $(X_t^{x,\phi})$ associated with a trading strategy ϕ and an initial capital x is equal to

$$X_t^{x,\phi} = x + \int_0^t \phi_s' \tilde{\mu}_s ds + \int_0^t \phi_s' \sigma_s dW_s + \int_0^t \phi_s' \beta_s dN_s\,.$$

Our goal is to solve the optimization problem for an agent who buys a contingent claim ξ

$$V(x,\xi) = \sup_{\phi\in\mathcal{A}(x)} \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{x,\phi} + \xi\big)\big)\big] = \exp(-\gamma x)V(0,\xi), \tag{2.5.11}$$

where $\mathcal{A}(x)$ is defined by:

Definition 2.5.1. The set of admissible trading strategies $\mathcal{A}(x)$ consists of all $\mathbb{G}$-predictable processes $\phi = (\phi_t)_{0\le t\le T}$, which satisfy $\int_0^T ||\phi_t'\sigma_t||^2 ds + \int_0^T |\phi_t'\beta_t|^2 \tilde{\lambda}_t dt < \infty,\ \mathbb{P}-a.s.$, and such that for any ϕ fixed and any $t \in [0,T]$, there exists a constant $K_{t,\phi}$ such that for any $s \in [t,T]$, we have $X_s^\phi - X_t^\phi \ge K_{t,\phi},\ \mathbb{P}-a.s.$

To solve this problem, it is sufficient to study the case $x = 0$. For the sake of brevity, we denote $\mathcal{A}$ instead of $\mathcal{A}(0)$ and X_t^ϕ (resp. $X_s^{t,\phi}$) instead of $X_t^{x,\phi}$ (resp. $X_s^{t,x,\phi}$). For that, we give a dynamic extension of the initial problem as in Section 2.4. For any initial time $t \in [0,T]$, we define the value function $J(t,\xi)$ by the following random variable

$$J(t,\xi) = \operatorname*{ess\,inf}_{\phi\in\mathcal{A}_t} \mathbb{E}\big[\exp\big(-\gamma\big(X_T^{t,\phi} + \xi\big)\big)\big|\mathcal{G}_t\big],$$

with $\mathcal{A}_t$ the admissible portfolio strategies set defined by:

Definition 2.5.2. The set of admissible trading strategies $\mathcal{A}_t$ consists of all $\mathbb{G}$-predictable processes $\phi = (\phi_s)_{t\le s\le T}$, which satisfy $\int_t^T ||\phi_s'\sigma_s||^2 ds + \int_t^T |\phi_s'\beta_s|^2 \tilde{\lambda}_s ds < \infty,\ \mathbb{P}-a.s.$, and such that for any ϕ fixed and any $s \in [t,T]$, there exists a constant $K_{s,\phi}$ such that for any $u \in [s,T]$, we have $X_u^\phi - X_s^\phi \ge K_{s,\phi},\ \mathbb{P}-a.s.$

We introduce the set $\mathcal{S}^{+,\infty}$ which corresponds to the set of positive $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded càd-làg processes on $[0,T]$.

By applying the results of the companion paper [91], we get the following characterizations of the value function:

Theorem 2.5.3. – *Let $(\bar{Y},\bar{Z},\bar{U})$ be the maximal solution* [2] *in $\mathcal{S}^{+,\infty}\times L^2(\bar{W})\times L^2(\bar{M})$ of*

$$\begin{cases} -d\bar{Y}_t = -\bar{Z}_t' d\bar{W}_t - \bar{U}_t' d\bar{M}_t + \underset{\phi\in\mathcal{A}}{\operatorname{ess\,inf}}\Big\{\dfrac{\gamma^2}{2}||\phi_t'\sigma_t||^2\bar{Y}_t - \gamma\phi_t'(\bar{Y}_t\tilde{\mu}_t + \sigma_t\bar{Z}_t) \\ \qquad\qquad - \big(\mathbb{1} - e^{-\gamma\phi_t'\beta_t}\big)(\bar{Y}_t\tilde{\lambda}_t + \tilde{\lambda}_t * \bar{U}_t)\Big\}dt, \\ \bar{Y}_T = \exp(-\gamma\xi), \end{cases} \tag{2.5.12}$$

then $\bar{Y}_t = \bar{J}(t,\xi),\ \mathbb{P}-a.s.$, for any $t\in[0,T]$.

– *$\bar{J}(t,\xi) = \lim_{n\to\infty}\downarrow \bar{J}^k(t,\xi)$, with $\bar{J}^k(t,\xi) = \operatorname{ess\,inf}_{\phi\in\mathcal{A}_t^k}\mathbb{E}[\exp(-\gamma(X_T^{t,\phi}+\xi))|\mathcal{G}_t]$, and $\mathcal{A}_t^k$ is the set of strategies of $\mathcal{A}_t$ uniformly bounded by k.*

– *Let $(\bar{Y}^k,\bar{Z}^k,\bar{U}^k)$ be the unique solution in $\mathcal{S}^2\times L^2(\bar{W})\times L^2(\bar{M})$ of the following BSDE*

$$\begin{cases} -d\bar{Y}_t^k = -\bar{Z}_t^{k'} d\bar{W}_t - \bar{U}_t^{k'} d\bar{M}_t + \underset{\phi\in\mathcal{A}^k}{\operatorname{ess\,inf}}\Big\{\dfrac{\gamma^2}{2}||\phi_t'\sigma_t||^2\bar{Y}_t^k - \gamma\phi_t'(\bar{Y}_t^k\tilde{\mu}_t + \sigma_t\bar{Z}_t^k) \\ \qquad\qquad - (\mathbb{1} - e^{-\gamma\phi_t'\beta_t})(\bar{Y}_t^k\tilde{\lambda}_t + \tilde{\lambda}_t * \bar{U}_t^k)\Big\}dt, \\ \bar{Y}_T^k = \exp(-\gamma\xi), \end{cases} \tag{2.5.13}$$

then $\bar{Y}_t^k = \bar{J}^k(t,\xi),\ \mathbb{P}-a.s.$, for any $t\in[0,T]$.

We can now define the indifference pricing of the contingent claim ξ. The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows: the issue at hand is to assess the value of some (defaultable) claim ξ as seen from the perspective of an investor who optimizes his behavior relative to some utility function, in our case we use the exponential utility function. The investor has two choices:

[2] That is for any solution $(\bar{J},\bar{Z},\bar{U})$ of BSDE (2.5.12) in $\mathcal{S}^{+,\infty}\times L^2(\bar{W})\times L^2(\bar{M})$, we have $\bar{J}_t \le J_t,\ \forall\, t\in[0,T],\ \mathbb{P}-a.s.$

– he only invests in the risk-free asset and in the risky assets, in this case the associated optimization problem is

$$V(x,0)=\sup_{\phi\in\mathcal{A}(x)}\mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{x,\phi}\big)\big)\big],$$

– he also invests in the contingent claim, whose price is $\bar{p}$ at 0, in this case the associated optimization problem is

$$V(x-\bar{p},\xi)=\sup_{\phi\in\mathcal{A}(x-p)}\mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{x-\bar{p},\phi}+\xi\big)\big)\big].$$

Definition 2.5.3. For a given initial capital x, the Hodges buying price of a defaultable claim ξ is the price $\bar{p}$ such that the investor's value functions are indifferent between holding and not holding the contingent claim, i.e.

$$V(x,0)=V(x-\bar{p},\xi).$$

The Hodges price $\bar{p}$ can be derived explicitly by applying the results of Theorem 2.5.3. If the agent buys the contingent claim at the price $\bar{p}$ and invests the rest of his wealth in the risk-free asset and in the risky assets, the value function is equal to

$$V(x-\bar{p},\xi)=-\exp(-\gamma(x-\bar{p}))\bar{J}(0,\xi).$$

If he invests all his wealth in the risk-free asset and in the risky assets, the value function is equal to

$$V(x,0)=-\exp(-\gamma x)\bar{J}(0,0).$$

Proposition 2.5.5. *The Hodges price for a contingent claim ξ is given by the formula*

$$\bar{p}=\frac{1}{\gamma}\ln\Big(\frac{\bar{J}(0,0)}{\bar{J}(0,\xi)}\Big),$$

where $\bar{J}(t,\xi)$ is the maximal solution to BSDE 2.5.12.

Proposition 2.5.6. *(Approximation of the indifference price) Suppose that the set of the admissible strategies is given by the bounded set $\mathcal{A}^k$. Let $\bar{p}^k$ be the indifference price defined by the same method. Then, we have*

$$\bar{p}^k=\frac{1}{\gamma}\ln\Big(\frac{\bar{J}^k(0,0)}{\bar{J}^k(0,\xi)}\Big),$$

where $\bar{J}^k(t,\xi)$ is defined in Theorem 2.5.3.

Also, we have

$$\bar{p}=\lim_{k\to\infty}\bar{p}^k.$$

Remark 2.5.2. That allows to approximate the indifference price by numerical computations. In particular, $\bar{J}^k(t,\xi)$ is the solution of a Lipschitz BSDE and the results of Bouchard and Elie [22] can be applied.

We assume that there are two kinds of agents in the market: the insider agents and the classical agents. We define the information price d for a contingent claim as the difference between the buying price for a classical agent and the buying price for an insider agent. The buying price, if the agent knows the full information, is defined by (see [91])

$$p = \frac{1}{\gamma}\ln\Big(\frac{J(0,0)}{J(0,\xi)}\Big),$$

where $(J(.,\xi), Z, U)$ is the maximal solution of BSDE (2.5.12) with $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$ replaced by $(W,\, M,\, \mu,\, \lambda)$.
Then, the benefit of an insider agent who has a full information is the information price

$$d = \bar{p} - p.$$

This price can be computed as the limit of the information prices $(d^k)_{k\in\mathbb{N}}$, where d^k is the information price if we restrict the admissible strategies set to the bounded set $\mathcal{A}^k$

$$d^k = \frac{1}{\gamma}\Big(\ln\Big(\frac{\bar{J}^k(0,0)}{J^k(0,0)}\Big) - \ln\Big(\frac{\bar{J}^k(0,\xi)}{J^k(0,\xi)}\Big)\Big),$$

where $(J^k(.,\xi), Z^k, U^k)$ is the solution of BSDE (2.5.13) with $(\bar{W},\, \bar{M},\, \tilde{\mu},\, \tilde{\lambda})$ replaced by $(W,\, M,\, \mu,\, \lambda)$.
Then, we have

$$d = \lim_{k\to\infty} d^k.$$

2.6 Appendix

2.6.1 Proof of Propositions 2.4.2 and 2.4.3

The proof of these propositions is based on the following lemma:

Lemma 2.6.1. *The set $\{J_t^\pi,\ \pi\in\mathcal{A}_t\}$ is stable by supremum for any $t\in[0,T]$, i.e. for any $\pi^1,\ \pi^2\in\mathcal{A}_t$, there exists $\pi\in\mathcal{A}_t$ such that $J_t^\pi = J_t^{\pi^1}\vee J_t^{\pi^2}$.*
Furthermore, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}\in\mathcal{A}_t$ for any $t\in[0,T]$, such that

$$J(t) = \lim_{n\to\infty}\uparrow J_t^{\pi^n},\ \mathbb{P}-a.s.$$

Proof. Let us introduce the set $E = \{J_t^{\pi^1} \geq J_t^{\pi^2}\}$ which belongs to $\mathcal{F}_t$. Let us define the strategy π by the formula $\pi_s = \pi_s^1 \mathbb{1}_E + \pi_s^2 \mathbb{1}_{E^c}$ for any $s \in [t,T]$. It is obvious that $\pi \in \mathcal{A}_t$. And by construction of π, it is clear that $J_t^{\pi} = J_t^{\pi^1} \vee J_t^{\pi^2}$.
The second part of the lemma follows by classical results on the essential supremum (see Neveu [100]). $\square$

We first prove that the process $((X_t^{\pi})^{\gamma} J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$. For that, it is sufficient to show for any $s \leq t$ that

$$\mathbb{E}\big[(X_t^{s,\pi})^{\gamma} J(t) \big| \mathcal{F}_s\big] \leq J(s),\ \mathbb{P}-a.s.$$

By Lemma 2.6.1, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{A}_t$ such that $J(t) = \lim \uparrow J_t^{\pi^n},\ \mathbb{P}-a.s.$ We define the strategy $\tilde{\pi}^n$ by $\tilde{\pi}_u^n = \pi_u \mathbb{1}_{[s,t]}(u) + \pi_u^n \mathbb{1}_{]t,T]}(u)$, which is clearly admissible. By the monotone convergence theorem and using the definition of $J(s)$, one can easily show that

$$\mathbb{E}\big[(X_t^{s,\pi})^{\gamma} J(t) \big| \mathcal{F}_s\big] = \lim_{n\to\infty} \uparrow \mathbb{E}\big[(X_T^{s,\tilde{\pi}^n})^{\gamma} \big| \mathcal{F}_s\big] \leq J(s),\ \mathbb{P}-a.s.$$

Hence, the process $((X_t^{\pi})^{\gamma} J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$.

Second, we prove that $(J(t))$ is the smallest process satisfying $((X_t^{\pi})^{\gamma} J(t))$ is a supermartingale for any $\pi \in \mathcal{A}$. For that, we suppose that $(\bar{J}_t)$ is an $\mathbb{F}$-adapted process such that $((X_t^{\pi})^{\gamma}(\bar{J}_t))$ is a supermartingale for any $\pi \in \mathcal{A}$ with the terminal condition $\bar{J}_T = 1$. Fix $t \in [0,T]$. For any $\pi \in \mathcal{A}$, we have $\mathbb{E}[(X_T^{\pi})^{\gamma} | \mathcal{F}_t] \leq (X_t^{\pi})^{\gamma} \bar{J}_t,\ \mathbb{P}-a.s.$ This inequality is equivalent to $\mathbb{E}[(X_T^{t,\pi})^{\gamma} | \mathcal{F}_t] \leq \bar{J}_t$. Which implies

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_t} \mathbb{E}\big[(X_T^{t,\pi})^{\gamma} \big| \mathcal{F}_t\big] \leq \bar{J}_t,\ \mathbb{P}-a.s.,$$

which clearly gives that $J_t \leq \bar{J}_t,\ \mathbb{P}-a.s.$

At last, we prove the optimality criterion, that is Proposition 2.4.3. Suppose that the strategy $\hat{\pi}$ is an optimal strategy, hence we have

$$J(0) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[\big(X_T^{\pi}\big)^{\gamma}\big] = \mathbb{E}\big[\big(X_T^{\hat{\pi}}\big)^{\gamma}\big].$$

As the process $((X_t^{\hat{\pi}})^{\gamma} J(t))$ is a supermartingale by Proposition 2.4.2 and that $J(0) = \mathbb{E}[(X_T^{\hat{\pi}})^{\gamma}]$, the process $((X_t^{\hat{\pi}})^{\gamma} J(t))$ is a martingale.
To show the converse, suppose that the process $((X_t^{\hat{\pi}})^{\gamma} J(t))$ is a martingale, then $\mathbb{E}[(X_T^{\hat{\pi}})^{\gamma}] = J(0)$. Moreover $\mathbb{E}[(X_t^{\pi})^{\gamma} J(t)] \leq J(0)$ for any $\pi \in \mathcal{A}$ by Proposition 2.4.2. Which implies that

$$J(0) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[\big(X_T^{\pi}\big)^{\gamma}\big] = \mathbb{E}\big[\big(X_T^{\hat{\pi}}\big)^{\gamma}\big].$$

2.6.2 Proof of Theorem 2.4.1

The proof of this theorem is based on Propositions 2.4.2 and 2.4.3, on Doob-Meyer's decomposition and on the martingale representation theorem.

Since the process (J_t) is a supermartingale, it can be written under the following form by using Doob-Meyer's decomposition (see [46]) and the martingale representation theorem

$$dJ_t = Z_t dW_t + U_t dM_t - dA_t, \tag{2.6.1}$$

with $Z \in L^2_{loc}(W)$, $U \in L^1_{loc}(M)$, and (A_t) is a nondecreasing $\mathbb{F}$-adapted process and $A_0 = 0$. From product rule, the derivative of process $((X_t^\pi)^\gamma J_t)$ can be written under the form

$$d((X_t^\pi)^\gamma J_t) = (X_{t^-}^\pi)^\gamma \big(dA_t^\pi + dM_t^\pi\big),$$

with $A_0^\pi = 0$ and

$$\begin{cases} dA_t^\pi = \Big[\gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \dfrac{\gamma(\gamma-1)}{2}|\pi_t\sigma_t|^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t + U_t)\Big]dt - dA_t, \\ dM_t^\pi = (\gamma\pi_t\sigma_t J_t + Z_t)dW_t + (U_t + ((1+\pi_t\beta_t)^\gamma - 1)(J_t+U_t))dM_t. \end{cases} \tag{2.6.2}$$

From Proposition 2.4.2, we have $dA_t^\pi \leq 0$ for any $\pi \in \mathcal{A}$, which implies

$$dA_t \geq \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}|\pi_t\sigma_t|^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t+U_t)\Big\}dt.$$

From [84], there exists an optimal strategy $\hat{\pi} \in \mathcal{A}$ to the optimization problem, and from Proposition 2.4.3, we get

$$dA_t = \Big[\gamma\hat{\pi}_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}|\hat{\pi}_t\sigma_t|^2 J_t + \lambda_t((1+\hat{\pi}_t\beta_t)^\gamma - 1)(J_t+U_t)\Big]dt.$$

Which imply that

$$dA_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big\{\gamma\pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}|\pi_t\sigma_t|^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t+U_t)\Big\}dt. \tag{2.6.3}$$

Therefore the process (J, Z, U) is a solution of BSDE (2.4.6).

We now prove that it is the minimal solution. Let $(\bar{J}, \bar{Z}, \bar{U})$ be a solution of BSDE (2.4.6). Let us prove that $((X_t^\pi)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \mathcal{A}$. From the product rule, we can write the derivative of this process under the form

$$d\left((X_t^\pi)^\gamma \bar{J}_t\right) = (X_{t^-}^\pi)^\gamma \left[d\bar{M}_t^\pi + d\bar{A}_t^\pi - d\bar{A}_t\right], \tag{2.6.4}$$

where $\bar{A}_t$ (resp. $\bar{M}_t^\pi$) is given by (2.6.3) (resp. 2.6.2) with (J, Z, U) replaced by $(\bar{J}, \bar{Z}, \bar{U})$, and $\bar{A}_0^\pi = 0$ and

$$d\bar{A}_t^\pi = \Big[\gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t)\Big]dt.$$

By integrating (2.6.4), we get

$$(X_t^\pi)^\gamma\bar{J}_t - \bar{J}_0 = \int_0^t (X_{s-}^\pi)^\gamma d\bar{M}_s^\pi - \int_0^t (X_s^\pi)^\gamma (d\bar{A}_s - d\bar{A}_s^\pi).$$

As $d\bar{A}_s \geq d\bar{A}_s^\pi$, we have $\int_0^t (X_{s-}^\pi)^\gamma d\bar{M}_s^\pi \geq (X_t^\pi)^\gamma\bar{J}_t - \bar{J}_0 \geq -\bar{J}_0$. It implies that $(\bar{M}_t^\pi)$ is a supermartingale, since it is a lower bounded local martingale. Hence, the process $((X_t^\pi)^\gamma\bar{J}_t)$ is a supermartingale for any $\pi \in \mathcal{A}$, because it is the sum of a supermartingale and a nonincreasing process. Proposition 2.4.2 implies that $J_t \leq \bar{J}_t,\ \forall t \in [0,T],\ \mathbb{P}-a.s.$, which ends this proof.

2.6.3 Proof of Theorem 2.4.2

We first remark that $(J^k(t))$ satisfies the following property:

Lemma 2.6.2. *The process $(J^k(t))$ is the smallest $\mathbb{F}$-adapted process such that $((X_t^\pi)^\gamma J^k(t))$ is a supermartingale for any $\pi \in \mathcal{A}^k$ with terminal condition $J^k(T) = 1$.*

To prove this lemma, we use exactly the same arguments as in the proof of Proposition 2.4.2, since Lemma 2.6.1 is still true with $\mathcal{A}_t^k$ instead of $\mathcal{A}_t$.

Fix $t \in [0,T]$. It is obvious with the definition of sets $\mathcal{A}_t$ and $\mathcal{A}_t^k$ that $\mathcal{A}_t^k \subset \mathcal{A}_t$ for each $k \in \mathbb{N}$, and hence

$$J^k(t) \leq J_t,\ \mathbb{P}-a.s. \tag{2.6.5}$$

Moreover, since $\mathcal{A}_t^k \subset \mathcal{A}_t^{k+1}$ for each $k \in \mathbb{N}$, it follows that the positive sequence $(J^k(t))_{k\in\mathbb{N}}$ is nondecreasing. Let us define the random variable

$$\tilde{J}(t) = \lim_{k\to\infty} \uparrow J^k(t),\ \mathbb{P}-a.s.$$

It is obvious that the process $\tilde{J}(t) \leq J_t,\ \mathbb{P}-a.s.$, from (2.6.5) and this holds for any $t \in [0,T]$. It remains to prove that $J_t \leq \tilde{J}(t),\ \mathbb{P}-a.s.$, for any $t \in [0,T]$. As in the proof of Theorem 5.1 of the companion paper [91], we first prove that the process $\tilde{J}(t^+)$ is càd-làg and satisfies $\tilde{J}(t^+) \leq \tilde{J}(t),\ \mathbb{P}-a.s.$ The process $((X_t^\pi)^\gamma\tilde{J}(t^+))$ is a supermartingale for any bounded strategy $\pi \in \mathcal{A}$. In the sequel, we shall denote $\bar{J}_t$ instead of $\tilde{J}(t^+)$. We now prove that $\bar{J}_t \geq J_t,\ \forall t \in [0,T],\ \mathbb{P}-a.s.$ Since $(\bar{J}_t)$ is a càd-làg supermartingale, it admits the following Doob-Meyer's decomposition

$$d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t - d\bar{A}_t,$$

with $\bar{Z} \in L^2_{loc}(W)$, $\bar{U} \in L^1_{loc}(M)$ and $(\bar{A}_t)$ is a nondecreasing $\mathbb{G}$-adapted process with $\bar{A}_0 = 0$. As before, we use the fact that the process $((X^\pi_t)^\gamma \bar{J}_t)$ is a supermartingale for any bounded strategy $\pi \in \mathcal{A}$ to give some conditions satisfied by the process $(\bar{A}_t)$. Let $\pi \in \mathcal{A}$ be a uniformly bounded strategy, the product rule gives

$$d((X^\pi_t)^\gamma \bar{J}_t) = (X^\pi_{t^-})^\gamma \big(d\bar{A}^\pi_t + d\bar{M}^\pi_t\big), \tag{2.6.6}$$

where $(\bar{A}^\pi_t)$ and $(\bar{M}^\pi_t)$ are given by (2.6.2) with $(J,\, Z,\, U,\, A)$ replaced by $(\bar{J},\, \bar{Z},\, \bar{U},\, \bar{A})$.

Let $\bar{\mathcal{A}}_t$ be the subset of uniformly bounded strategies of $\mathcal{A}_t$. Since the process $((X^\pi_t)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \bar{\mathcal{A}}$, we have

$$d\bar{A}_t \geq \operatorname*{ess\,sup}_{\pi \in \bar{\mathcal{A}}} \Big\{ \gamma \pi_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2} |\pi_t \sigma_t|^2 \bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t) \Big\} dt. \tag{2.6.7}$$

It is not possible to give an exact expression of $\bar{A}_t$ as in the previous proof, because we do not know if $\hat{\pi} \in \bar{\mathcal{A}}$. But this inequality is sufficient for the proof. Now, the following equality holds $dt \otimes d\mathbb{P}$ $a.s.$

$$\begin{aligned} &\operatorname*{ess\,sup}_{\pi \in \bar{\mathcal{A}}} \Big\{ \gamma \pi_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2} |\pi_t \sigma_t|^2 \bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t) \Big\} = \\ &\quad \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \gamma \pi_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2} |\pi_t \sigma_t|^2 \bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t) \Big\}. \end{aligned} \tag{2.6.8}$$

We now want to show that $((X^\pi_t)^\gamma \bar{J}_t)$ is a supermartingale for any $\pi \in \mathcal{A}$. Fix $\pi \in \mathcal{A}$ (not necessarily uniformly bounded), we get

$$(X^\pi_t)^\gamma \bar{J}_t - \bar{J}_0 = \int_0^t (X^\pi_{s^-})^\gamma d\bar{M}^\pi_s + \int_0^t (X^\pi_s)^\gamma d\bar{A}^\pi_s,$$

with $(\bar{A}^\pi_t)$ and $(\bar{M}^\pi_t)$ given by (2.6.2) with $(J,\, Z,\, U,\, A)$ replaced by $(\bar{J},\, \bar{Z},\, \bar{U},\, \bar{A})$.

Inequality (2.6.7) and equality (2.6.8) imply that $d\bar{A}^\pi_t \leq 0$, $\mathbb{P}-a.s.$ Therefore, we have

$$\int_0^t (X^\pi_{s^-})^\gamma d\bar{M}^\pi_s \geq (X^\pi_t)^\gamma \bar{J}_t - \bar{J}_0 \geq -\bar{J}_0.$$

Thus, $(\bar{M}^\pi_t)$ is a supermartingale, since it is a lower bounded local martingale. As $(\bar{M}^\pi_t)$ is a supermartingale and $(\bar{A}^\pi_t)$ is nonincreasing, the process $((X^\pi_t)^\gamma \bar{J}_t)$ is a supermartingale, and this holds for any $\pi \in \mathcal{A}$. Since (J_t) is the smallest process (see Proposition 2.4.2) satisfying these properties, we have $J_t \leq \bar{J}_t$, $\mathbb{P}-a.s.$ Which ends the proof.

2.6.4 Proof of Lemma 2.5.3

First, recall Bayes formula: for all $t \in [0,T]$ and $X \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$, one has

$$\mathbb{E}\big[X \big| \mathcal{G}_t\big] = \frac{\mathbb{E}_{\mathbb{Q}}\big[\Lambda_t X \big| \mathcal{G}_t\big]}{\tilde{\Lambda}_t}. \tag{2.6.9}$$

Let (ξ_t) be the optional projection of the $\mathbb{P}$-martingale (L_t) to $\mathbb{G}$, so

$$\xi_t = \mathbb{E}\big[L_t \big| \mathcal{G}_t\big].$$

By applying relation (2.6.9) to $X = L_t$, we immediately obtain $\xi_t = 1/\tilde{\Lambda}_t$ and thus

$$\xi_t = \exp\Big(- \int_0^t \tilde{\rho}_s' d\bar{W}_s - \frac{1}{2}\int_0^t ||\tilde{\rho}_s||^2 ds\Big).$$

Let (m_t) be a $(\mathbb{P}, \mathbb{G})$-local martingale. From Bayes rule, the process $(\tilde{m}_t)$ given by

$$\tilde{m}_t = m_t \xi_t^{-1}, \quad 0 \leq t \leq T,$$

is a $(\mathbb{Q}, \mathbb{G})$-local martingale. From Remark 2.5.1 and Lemma 2.4.1, there exists a couple of processes $(\tilde{a}_t, \tilde{b}_t)$ with $\tilde{a} \in L^2_{loc}(\tilde{W})$ and $\tilde{b} \in L^1_{loc}(\bar{M})$ such that

$$\tilde{m}_t = \int_0^t \tilde{a}_s' d\tilde{W}_s + \int_0^t \tilde{b}_s' d\bar{M}_s, \quad 0 \leq t \leq T.$$

By Itô's formula applied to $m_t = \tilde{m}_t \xi_t$, definition of $(\bar{W}_t)$ and $(\bar{M}_t)$ (see (2.5.6)), we obtain that

$$m_t = \int_0^t a_s' d\bar{W}_s + \int_0^t b_s' d\bar{M}_s,$$

with $a_t = \xi_t \tilde{a}_t - \tilde{m}_t \xi_t \tilde{\rho}_t$ and $b_t = \xi_{t^-} \tilde{b}_t$.

Part II

PROGRESSIVE ENLARGEMENT OF FILTRATIONS AND BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

Chapter 3

Progressive enlargement of filtrations and Backward SDEs with jumps

Joint work with Idris Kharroubi.

Abstract: This work deals with backward stochastic differential equation (BSDE) with random default times, and their applications to default risk. We show that these BSDEs are linked with Brownian BSDEs through the decomposition of processes with respect to the progressive enlargement of filtrations. We show that the BSDEs have solutions if the associated Brownian BSDEs have solutions. We also give a uniqueness theorem, and a new Feynman-Kac formula for integral partial differential equations. As applications, we study the pricing and the hedging of a European option in a complete market, then the indifference pricing of defaultable claims in an incomplete market.

Keywords: Backward SDE, multiple random default times, progressive enlargement of filtrations, decomposition in the reference filtration, uniqueness theorem, Feynman-Kac formula, indifference pricing.

3.1 Introduction

In recent years, credit risk has come out to be one of most fundamental financial risk. The most extensively studied form of credit risk is the default risk. Many people, such as Bielecki, Jarrow, Jeanblanc, Pham, Rutkowski ([16, 15, 70, 72, 76, 111]) and many others, have worked on this subject. In several papers (see for example Ankirchner *et al.* [3], Bielecki and Jeanblanc [17], Lim and Quenez [91] and Peng and Xu [109]), related to this topic, backward stochastic differential equations (BSDEs) with random default times have appeared. Unfortunately, the results relative to these latter BSDEs are far from being as numerous as for Brownian BSDEs. In particular, there is not any general result on the existence of solution to quadratic BSDEs, except Ankirchner *et al.* [3], where the assumptions on the driver are strong. In this paper, we study BSDEs with random default times. We give an existence and uniqueness result for the solutions to BSDEs, in particular for quadratic BSDEs.

A standard approach of credit risk modeling is based on the powerful technique of filtration enlargement, by making the distinction between the filtration $\mathbb{F}$ generated by the Brownian motion, and its smallest extension $\mathbb{G}$ that turns default times into $\mathbb{G}$-stopping times. This kind of filtration enlargement has been referred to as progressive enlargement of filtrations. This field of enlargement of filtrations is a traditional subject in probability theory initiated by fundamental works of the French school in the 80s, see e.g. Jeulin [74], Jeulin and Yor [75], and Jacod [69]. For an overview of applications of progressive enlargement of filtrations on credit risk, we refer to the books of Duffie and Singleton [50], of Bielecki and Rutkowski [16], or the lecture notes of Bielecki *et al.* [15]. A classical assumption in the enlargement of filtrations is the stability of the class of semimartingales, usually called **(H')** hypothesis, and meaning that any $\mathbb{F}$-semimartingale remains a $\mathbb{G}$-semimartingale. This assumption is a fundamental property both in probability and finance where it is closely related to the absence of arbitrage.

The purpose of this paper is to combine both Brownian BSDEs and progressive enlargement of filtrations in view of giving results about BSDEs with random default times. We consider a progressive enlargement with multiple random times and associated marks. These marks represent for example the name of the firm which defaults or the jump sizes of asset values. Our approach consists in using the recent results of Pham [111] on the decomposition of predictable and optional processes with respect to the progressive enlargement filtrations to decompose a BSDE with random default times into a sequence of Brownian BSDEs. By combining the solutions of Brownian BSDEs, we get a solution to the BSDE with random default times. Using this method, we get in particular an existence result for quadratic BSDEs. This approach also allows to obtain a uniqueness theorem which is based on a comparison theorem. The same technique can be applied to the integral partial differential equations, and this gives a decomposition of Feynman-Kac formula for these equations. This decomposition of Feynman-Kac formula is written as a recursive system of partial differential equations. We illustrate our methodology with two financial applications in default risk management: the pricing and the hedging of a European option in a complete market, and the problem of indifference pricing of defaultable claims in an incomplete market. A similar problem (without marks) was recently considered in Ankirchner *et al.* [3] and Lim and Quenez [91].

The paper is organized as follows. The next section presents the general framework of progressive enlargement of filtrations with successive random times and marks, and states the decomposition result for $\mathbb{G}$-predictable and optional processes. In Section 3, we use this decomposition to make a link between Brownian BSDEs and BSDEs with random default times. That allows to give an existence and uniqueness result. We also give a first example with the pricing and the hedging of a European option in a complete market. In Section 4, we give a new Feynman-Kac formula for IPDE, which could be regarded as a decomposition of Feynman-Kac formula given by Barles, Buckdahn and Pardoux [7]. Finally, in Section 5, we use the results of existence and uniqueness to solve the problem of maximization of exponential utility and we study the indifference pricing of defaultable claims in an incomplete market.

3.2 Progressive enlargement of filtrations with successive random times and marks

We fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and we start with a reference filtration $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ satisfying the usual conditions ($\mathcal{F}_0$ contains the $\mathbb{P}$-null sets and $\mathbb{F}$ is right continuous: $\mathcal{F}_t = \mathcal{F}_{t^+} := \cap_{s>t}\mathcal{F}_s$). We consider a finite sequence $(\tau_k, \zeta_k)_{1\leq k\leq n}$ where

- $(\tau_k)_{1\leq k\leq n}$ is a sequence of random times (i.e. nonnegative $\mathcal{G}$-random variables),
- $(\zeta_k)_{1\leq k\leq n}$ is a sequence of random marks valued in some Borel subset E of $\mathbb{R}^m$.

We denote by μ the random measure associated with the sequence $(\tau_k, \zeta_k)_{1\leq k\leq n}$:

$$\mu([0,t]\times B) \;=\; \sum_{k=1}^{n} \mathbb{1}_{\{\tau_k\leq t,\, \zeta_k\in B\}}\ .$$

For each $k = 1, \ldots, n$, we consider $\mathbb{D}^k = (\mathcal{D}^k_t)_{0\leq t\leq T}$ the smallest right-continuous filtration for which τ_k is a stopping time and ζ_k is $\mathcal{D}^k_{\tau_k}$-measurable. $\mathbb{D}^k$ is then given by $\mathcal{D}^k_t = \tilde{\mathcal{D}}^k_{t^+}$, where $\tilde{\mathcal{D}}^k_t = \sigma(\mathbb{1}_{\tau_k\leq s}, s\leq t) \vee \sigma(\zeta_k\mathbb{1}_{\tau_k\leq s}, s\leq t)$. The global information is then defined by the progressive enlargement $\mathbb{G} = \{\mathcal{G}_t,\ 0\leq t\leq T\}$ of the initial filtration $\mathbb{F}$ where $\mathbb{G} := \mathbb{F}\vee\mathbb{D}^1\vee\ldots\vee\mathbb{D}^n$. The filtration $\mathbb{G} = \{\mathcal{G}_t,\ 0\leq t\leq T\}$ is the smallest filtration containing $\mathbb{F}$, and such that for each $k = 1, \ldots, n$, τ_k is a $\mathbb{G}$-stopping time, and ζ_k is $\mathcal{G}_{\tau_k}$-measurable. We first remark that an enlargement of the free default filtration $\mathbb{F}$ by unordered random times is equivalent to the enlargement with the associated ordered random times and random marks. Indeed, let $(\hat{\tau}_1, \ldots, \hat{\tau}_n)$ be the order statistics of $(\tau_1, \ldots, \tau_n)$. Call $(\zeta_1, \ldots, \zeta_n)$ the corresponding permutation which satisfies

$$(\hat{\tau}_1, \ldots, \hat{\tau}_n) = (\tau_{\zeta_1}, \ldots, \tau_{\zeta_n}), \ \mathbb{P} - a.s. \tag{3.2.1}$$

Hence, for simplicity of presentation, we only consider in the sequel the case where the random times are ordered, i.e. $\tau_1 \leq \ldots \leq \tau_n$, and so valued in Δ_n on $\{\tau_n < \infty\}$, with

$$\Delta_k := \big\{(\theta_1, \ldots, \theta_k) \in (\mathbb{R}_+)^k \ : \ \theta_1 \leq \ldots \leq \theta_k\big\}, \ 1 \leq k \leq n.$$

We introduce some notations used throughout the paper:

- $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets on $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes. We also denote by $\mathcal{P}_\mathbb{F}$ (resp. $\mathcal{P}_\mathbb{G}$) the set of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable processes, i.e. $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$)-measurable processes,

- $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-optional measurable subsets on $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the right-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes. We also denote by $\mathcal{O}_\mathbb{F}$ (resp. $\mathcal{O}_\mathbb{G}$) the set of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable processes, i.e. $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$)-measurable processes,

- for each $k = 1, \ldots, n$, we denote by $\mathcal{P}^k_\mathbb{F}(\Delta_k, E^k)$ (resp. $\mathcal{O}^k_\mathbb{F}(\Delta_k, E^k)$) the set of indexed processes $Y^k(.)$ such that the map

  $$(\omega, t, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k) \quad \mapsto \quad Y^k_t(\omega, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$$

  is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$ (resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$)-measurable,

- for $\theta = (\theta_1, \ldots, \theta_n) \in \Delta_n$ and $e = (e_1, \ldots, e_n) \in E^n$ we denote by

  $$\theta_{(k)} \quad = \quad (\theta_1, \ldots, \theta_k) \quad \text{and} \quad e_{(k)} \ = \ (e_1, \ldots, e_k), \ k = 1, \ldots, n.$$

In the sequel, we will denote $Y^k_t(\omega, \theta_{(k)}, e_{(k)})$ instead of $Y^k_t(\omega, \theta_1, \ldots, \theta_k, e_1, \ldots, e_k)$.

The following result, given by Pham [111], provides the basic decomposition of predictable and optional processes with respect to this progressive enlargement of filtrations:

Lemma 3.2.1. – *Any $\mathbb{G}$-predictable process $Y = (Y_t)_{0 \leq t \leq T}$ is represented as*

$$Y_t = Y^0_t \mathbb{1}_{t \leq \tau_1} + \sum_{k=1}^{n-1} Y^k_t(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k < t \leq \tau_{k+1}} + Y^n_t(\tau_{(n)}, \zeta_{(n)}) \mathbb{1}_{\tau_n < t}, \tag{3.2.2}$$

for all $0 \leq t \leq T$, where $Y^0 \in \mathcal{P}_\mathbb{F}$, and $Y^k \in \mathcal{P}^k_\mathbb{F}(\Delta_k, E^k)$, for $k = 1, \ldots, n$.

– *Any $\mathbb{G}$-optional process $Y = (Y_t)_{0\le t\le T}$ is represented as*

$$Y_t = Y_t^0 \mathbb{1}_{t<\tau_1} + \sum_{k=1}^{n-1} Y_t^k(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \le t < \tau_{k+1}} + Y_t^n(\tau_{(n)}, \zeta_{(n)}) \mathbb{1}_{\tau_n \le t}, \tag{3.2.3}$$

for all $0 \le t \le T$, where $Y^0 \in \mathcal{O}_{\mathbb{F}}$, and $Y^k \in \mathcal{O}_{\mathbb{F}}^k(\Delta_k, E^k)$, for $k = 1, \ldots, n$.

In view of the decomposition (3.2.2) or (3.2.3), we can then identify any $Y \in \mathcal{P}_{\mathbb{G}}$ (resp. $\mathcal{O}_{\mathbb{G}}$) with an $n+1$-tuple $(Y^0, \ldots, Y^n) \in \mathcal{P}_{\mathbb{F}} \times \ldots \times \mathcal{P}_{\mathbb{F}}^n(\Delta_n, E^n)$ (resp. $\mathcal{O}_{\mathbb{F}} \times \ldots \times \mathcal{O}_{\mathbb{F}}^n(\Delta_n, E^n)$).

We shall make in the sequel the standing assumption of the semimartingale invariance property, also called **(H')** hypothesis, i.e. any $\mathbb{F}$-semimartingale remains a $\mathbb{G}$-semimartingale. This result is related in finance to no-arbitrage conditions, and is thus also a desirable property from an economical viewpoint.

We now introduce a density assumption on the random times and their associated jumps by assuming that the distribution of $(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n)$ is absolutely continuous with respect to a positive measure $d\theta\, \eta(de)$ on $\mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)$, with η a measure on $\mathcal{B}(E^n)$. More precisely, we assume that there exists a $\mathcal{B}(\Delta_n) \otimes \mathcal{B}(E^n)-$measurable map γ such that

$$\textbf{(DH)} \qquad \mathbb{P}[(\tau_1, \ldots, \tau_n, \zeta_1, \ldots, \zeta_n) \in d\theta de] \;=\; \gamma(\theta_1, \ldots, \theta_n, e_1, \ldots, e_n) d\theta_1 \ldots d\theta_n \eta(de)\ .$$

The assumption of a density w.r.t the Lebesgue measure for λ is made for simplicity, and includes usual cases of application.

Remark 3.2.1. *If the unordered times $(\tau_1, \ldots, \tau_n)$ satisfy the density assumption:*

$$\mathbb{E}[f(\tau_1, \ldots, \tau_n)] \;=\; \int_{\mathbb{R}_+^n} f(s)\gamma(s) ds\ \mathbb{P} - a.s.,$$

then, the ordered times (with marks) $(\hat{\tau}_1, \ldots, \hat{\tau}_n, \zeta_1, \ldots, \zeta_n)$ satisfy also the density assumption with density [1]

$$\mathbb{E}[f(\hat{\tau}_1, \ldots, \hat{\tau}_n, \zeta_1, \ldots, \zeta_n)] \;=\; \int_{\Delta_n} \sum_{\sigma \in \mathfrak{S}_n} f(s, \sigma)\hat{\gamma}(s, \sigma) ds,\ \mathbb{P} - a.s.,$$

where $\hat{\gamma}$ is defined by

$$\hat{\gamma}(\omega, s_1, \ldots, s_n, \sigma) \;=\; \gamma(\omega, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(n)}).$$

[1] $\mathfrak{S}_n$ is the set of permutations of $\{1, \ldots, n\}$.

3.3 Decomposition of BSDEs with jumps

In this section, we use the previous decomposition results to solve BSDEs with jumps. We use a similar approach to Ankirchner *et al.* [3]: one can explicitly construct a solution by combining solutions of Brownian BSDEs. But contrary to them, we suppose that there exist n default times and n random marks. Our assumptions on the driver are also weaker.

We first introduce some notations:

– $\mathcal{S}^\infty_{\mathbb{G}}$ (resp. $\mathcal{S}^\infty_{\mathbb{F}}$) is the set of processes $Y \in \mathcal{O}_{\mathbb{G}}$ (resp. $Y \in \mathcal{O}_{\mathbb{F}}$) essentially bounded:

$$\|Y\|_{\mathcal{S}^\infty_{\mathbb{G}}} \text{ (resp. } \|Y\|_{\mathcal{S}^\infty_{\mathbb{F}}}) \quad := \quad \operatorname*{ess\,sup}_{t\in[0,T]} |Y_t| \ < \ \infty \, .$$

– $L^2_{\mathbb{G}}(W)$ (resp. $L^2_{\mathbb{F}}(W)$) is the set of $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes Z such that

$$||Z||_{L^2_{\mathbb{G}}(W)} \text{ (resp. } ||Z||_{L^2_{\mathbb{F}}(W)}) \quad := \quad \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] \ < \ \infty \, .$$

– $L^2(\mu)$ is the set of $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(E)$-measurable processes U such that

$$\|U\|_{L^2(\mu)} \quad = \quad \mathbb{E}\Big[\int_0^T \int_E |U_s(e)|^2 \mu(de, ds)\Big] \ < \ \infty \, .$$

Throughout this section, we will consider one dimensional BSDEs of the form

$$Y_t \ = \ \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds), \ 0 \le t \le T. \quad (3.3.1)$$

3.3.1 Existence of a solution

We first define what is a solution to BSDE (3.3.1):

Definition 3.3.1. A solution in $\mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}(W) \times L^2(\mu)$ to BSDE (3.3.1) is a triple of processes (Y, Z, U) in $\mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}(W) \times L^2(\mu)$ satisfying the equality

$$Y_t \quad = \quad \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) \mu(de, ds),$$

for all $t \in [0, T]$, $\mathbb{P} - a.s.$ [2]

Using the decomposition of Lemma 3.2.1, we link these BSDEs with Brownian BSDEs. For this purpose we first introduce the basic decompositions of ξ and f:

$$\xi \quad = \quad \xi^0 \mathbb{1}_{T \le \tau_1} + \sum_{k=1}^n \xi^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) \mathbb{1}_{\tau_k < T \le \tau_{k+1}}, \quad (3.3.2)$$

[2] The symbol $\int_s^t$ stands for the integral on the interval $(s, t]$ for all $s, t \in \mathbb{R}_+$.

where ξ^0 is $\mathcal{F}_T$-measurable and ξ^k is $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable for each $k = 1, \dots, n$, with the convention that $\tau_{n+1} = \infty$, and

$$f(t,y,z,u) = f^0(t,y,z,u)\mathbb{1}_{t<\tau_1} + \sum_{k=1}^{n} f^k(t,y,z,u,\tau_1,\dots,\tau_k,\zeta_1,\dots,\zeta_k)\mathbb{1}_{\tau_k \le t < \tau_{k+1}}, \quad (3.3.3)$$

where f^0 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^E)$-measurable and f^k is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^E) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable for each $k = 1, \dots, n$. To alleviate notation, we shall often denote ξ^k and $f^k(t,y,z,u)$ instead of $\xi^k(\tau_{(k)}, \zeta_{(k)})$ and $f^k(t,y,z,u,\tau_{(k)},\zeta_{(k)})$, and $Y^k_t(t,e)$ instead of $Y^k_t(\theta_{(k-1)}, t, e_{(k-1)}, e)$.

The link between Brownian BSDEs and BSDEs with jumps is given by the following result.

Theorem 3.3.1. *Assume that for all $(\theta, e) \in \Delta_n \times E^n$, the Brownian BSDE*

$$\begin{aligned} Y^n_t(\theta,e) &= \xi^n(\theta,e) + \int_t^T f^n\Big(s, Y^n_s(\theta,e), Z^n_s(\theta,e), 0, \theta, e\Big) ds \\ &\quad - \int_t^T Z^n_s(\theta,e) dW_s , \quad 0 \le t \le T , \end{aligned} \quad (3.3.4)$$

admits a solution $(Y^n(\theta,e), Z^n(\theta,e)) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}(W)$, and that for each $k = 0, \dots, n-1$, the BSDE

$$\begin{aligned} Y^k_t(\theta_{(k)}, e_{(k)}) &= \xi^k(\theta_{(k)}, e_{(k)}) + \int_t^T f^k\Big(s, Y^k_s(\theta_{(k)}, e_{(k)}), Z^k_s(\theta_{(k)}, e_{(k)}), \\ &\quad Y^{k+1}_s(\theta_{(k)}, s, e_{(k)}, .) - Y^k_s(\theta_{(k)}, e_{(k)})\Big) ds \\ &\quad - \int_t^T Z^k_s(\theta_{(k)}, e_{(k)}) dW_s , \quad 0 \le t \le T , \end{aligned} \quad (3.3.5)$$

admits a solution $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)})) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}(W)$. Assume moreover that each Y^k (resp. Z^k) is $\mathcal{O}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable (resp. $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable).

If all these solutions satisfy

$$\sup_{(k,\theta,e) \in \{0,\dots,n\} \times \Delta_n \times E^n} ||Y^k(\theta_{(k)}, e_{(k)})||_{\mathcal{S}^\infty_{\mathbb{F}}} < \infty ,$$

and

$$\int_{\Delta_n \times E^n} \mathbb{E}\Big[\int_0^{\theta_1 \wedge T} |Z^0_s|^2 ds + \sum_{k=1}^n \int_{\theta_k \wedge T}^{\theta_{k+1} \wedge T} |Z^k_s(\theta_{(k)}, e_{(k)})|^2 ds\Big] \gamma(\theta, e) d\theta \, \eta(de) < \infty ,$$

then, BSDE (3.3.1) *admits a solution* $(Y, Z, U) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}(W) \times L^2(\mu)$ *given by*

$$\left\{\begin{array}{rcl} Y_t &=& Y^0_t \mathbb{1}_{t<\tau_1} + \sum_{k=1}^{n} Y^k_t(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \leq t < \tau_{k+1}}, \\ Z_t &=& Z^0_t \mathbb{1}_{t\leq\tau_1} + \sum_{k=1}^{n} Z^k_t(\tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k < t \leq \tau_{k+1}}, \\ U_t(.) &=& U^0_t(.) \mathbb{1}_{t\leq\tau_1} + \sum_{k=1}^{n-1} U^k_t(\tau_{(k)}, \zeta_{(k)}, .) \mathbb{1}_{\tau_k < t \leq \tau_{k+1}}, \end{array}\right. \tag{3.3.6}$$

with $U^0_t(.) = Y^1_t(t, .) - Y^0_t$ *and* $U^k_t(\tau_{(k)}, \zeta_{(k)}, .) = Y^{k+1}_t(\tau_{(k)}, t, \zeta_{(k)}, .) - Y^k_t(\tau_{(k)}, \zeta_{(k)})$ *for each* $k = 1, \ldots, n-1$.

Proof. For the simplicity of notation, we shall omit the dependence of processes on $(\theta_{(k)}, e_{(k)})$. We prove that (Y, Z, U) defined by (3.3.6) satisfied the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds), \quad 0 \leq t \leq T.$$

We distinguish three cases.

Case 1: there are n defaults before t. Hence, $\tau_n \leq t$ and from (3.3.6) we get $Y_t = Y^n_t$. Using BSDE (3.3.4), we can see that

$$Y^n_t = \xi^n + \int_t^T f^n(s, Y^n_s, Z^n_s, 0)ds - \int_t^T Z^n_s dW_s .$$

Since $\tau_n < T$, we have $\xi^n = \xi$ from (3.3.2). And in the same way, we have $Y_s = Y^n_s$, $Z_s = Z^n_s$, $U_s = 0$ for all $s \in (t, T]$ from (3.3.6), and $f^n(s, Y^n_s, Z^n_s, 0) = f(s, Y_s, Z_s, U_s)$ for all $s \in (t, T]$ from (3.3.3), then

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds) .$$

Case 2: there is no default after t and i defaults before t ($i < n$). Hence, $Y_t = Y^i_t$, and using BSDE (3.3.5), we can see that

$$Y^i_t = \xi^i + \int_t^T f^i(s, Y^i_s, Z^i_s, Y^{i+1}_s(s, .) - Y^i_s)ds - \int_t^T Z^i_s dW_s .$$

Since there is no default after t, we have $Y_s = Y^i_s$, $Z_s = Z^i_s$, $U^i_s(.) = Y^{i+1}_s(s, .) - Y^i_s$, $\xi = \xi^i$ and $f^i(s, Y^i_s, Z^i_s, U^i_s) = f(s, Y_s, Z_s, U_s)$ for all $s \in (t, T]$, and moreover $\int_t^T \int_E U_s(e)\mu(de, ds) = 0$ because there is no default on $(t, T]$, thus we have

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds) .$$

Case 3: there are k defaults after t and i defaults before t. Since τ_{k+i} is the last default time, we have $Y_{\tau_{k+i}} = Y^{k+i}_{\tau_{k+i}}$. And since Y^{k+i} is a solution of BSDE (3.3.4) if $k+i=n$ or (3.3.5) if $k+i<n$, we can see in the first case that

$$Y^{k+i}_{\tau_{k+i}} = \xi^{k+i} + \int_{\tau_{k+i}}^T f^{k+i}(s, Y^{k+i}_s, Z^{k+i}_s, 0)ds - \int_{\tau_{k+i}}^T Z^{k+i}_s dW_s \ ,$$

or in the second case that

$$Y^{k+i}_{\tau_{k+i}} = \xi^{k+i} + \int_{\tau_{k+i}}^T f^{k+i}(s, Y^{k+i}_s, Z^{k+i}_s, Y^{k+i+1}_s(s,.) - Y^{k+i}_s)ds - \int_{\tau_{k+i}}^T Z^{k+i}_s dW_s \ .$$

With the same techniques, we get for $Y_{\tau_{k+i-1}}$

$$\begin{aligned} Y^{k+i-1}_{\tau_{k+i-1}} &= Y^{k+i-1}_{\tau_{k+i}} + \int_{\tau_{k+i-1}}^{\tau_{k+i}} f^{k+i-1}(s, Y^{k+i-1}_s, Z^{k+i-1}_s, Y^{k+i}_s(s,.) - Y^{k+i-1}_s)ds \\ &\quad - \int_{\tau_{k+i-1}}^{\tau_{k+i}} Z^{k+i-1}_s dW_s \ . \end{aligned}$$

Using the equality $Y^{k+i-1}_{\tau_{k+i}} = Y^{k+i}_{\tau_{k+i}}(\tau_{k+i}, .) - \int_{\tau_{k+i-1}}^{\tau_{k+i}} \int_E U^{k+i-1}_s(e)\mu(de, ds)$, we get

$$\begin{aligned} Y^{k+i-1}_{\tau_{k+i-1}} &= Y^{k+i}_{\tau_{k+i}}(\tau_{k+i}, .) - \int_{\tau_{k+i-1}}^{\tau_{k+i}} \int_E U^{k+i-1}_s(e)\mu(de, ds) - \int_{\tau_{k+i-1}}^{\tau_{k+i}} Z^{k+i-1}_s dW_s \\ &\quad + \int_{\tau_{k+i-1}}^{\tau_{k+i}} f^{k+i-1}(s, Y^{k+i-1}_s, Z^{k+i-1}_s, Y^{k+i}_s(s,.) - Y^{k+i-1}_s)ds \ . \end{aligned}$$

By iteration until τ_{i+1}, we get

$$\begin{aligned} Y^{i+1}_{\tau_{i+1}} &= Y^{i+2}_{\tau_{i+2}}(\tau_{i+2}, .) - \int_{\tau_{i+1}}^{\tau_{i+2}} \int_E U^{i+1}_s(e)\mu(de, ds) - \int_{\tau_{i+1}}^{\tau_{i+2}} Z^{i+1}_s dW_s \\ &\quad + \int_{\tau_{i+1}}^{\tau_{i+2}} f^{i+1}(s, Y^{i+1}_s, Z^{i+1}_s, Y^{i+2}_s(s,.) - Y^{i+1}_s)ds \ . \end{aligned}$$

Since $\tau_i \le t < \tau_{i+1}$, we have $Y_t = Y^i_t$. Using that Y^i is a solution of (3.3.5), we get

$$Y^i_t = Y^i_{\tau_{i+1}} + \int_t^{\tau_{i+1}} f^i(s, Y^i_s, Z^i_s, Y^{i+1}_s(s,.) - Y^i_s)ds - \int_t^{\tau_{i+1}} Z^i_s dW_s \ .$$

Using the equality $Y^i_{\tau_{i+1}} = Y^{i+1}_{\tau_{i+1}}(\tau_{i+1}, .) - \int_t^{\tau_{i+1}} \int_E U^i_s(e)\mu(de, ds)$, we get

$$\begin{aligned} Y^i_t &= Y^{i+1}_{\tau_{i+1}}(\tau_{i+1}, .) + \int_t^{\tau_{i+1}} f^i(s, Y^i_s, Z^i_s, Y^{i+1}_s(s,.) - Y^i_s)ds \\ &\quad - \int_t^{\tau_{i+1}} Z^i_s dW_s - \int_t^{\tau_{i+1}} \int_E U^i_s(e)\mu(de, ds) \ . \end{aligned}$$

If we sum all these equations, and with the expressions of Y, Z and U given by (3.3.6), and the decompositions of ξ and f given by (3.3.2) and (3.3.3), we get

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds) \ .$$

We now prove that the solution satisfies the integrability conditions. By definition of Y, we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in[0,T]} |Y_t| &\leq \operatorname*{ess\,sup}_{(t,\theta,e)\in[0,T]\times\Delta_n\times E^n} \Big| Y_t^0 \mathbb{1}_{t<\theta_1} + \sum_{k=1}^n Y_t^k(\theta_{(k)}, e_{(k)}) \mathbb{1}_{\theta_k \leq t < \theta_{k+1}} \Big| , \\
&\leq \operatorname*{ess\,sup}_{(t,\theta,e)\in[0,T]\times\Delta_n\times E^n} \Big\{ \Big| Y_t^0 \Big| \mathbb{1}_{t<\theta_1} + \sum_{k=1}^n \Big| Y_t^k(\theta_{(k)}, e_{(k)}) \Big| \mathbb{1}_{\theta_k \leq t < \theta_{k+1}} \Big\} , \\
&\leq \operatorname*{ess\,sup}_{t\in[0,T]} |Y_t^0| + \sum_{k=1}^n \operatorname*{ess\,sup}_{(t,\theta_{(k)},e_{(k)})\in[0,T]\times\Delta_k\times E^k} |Y_t^k(\theta_{(k)}, e_{(k)})| .
\end{aligned}$$

Thus, $Y \in \mathcal{S}^\infty_{\mathbb{G}}$ since the processes $(Y^k(\theta_{(k)}, e_{(k)}))_{k\in\{0,\dots,n\}}$ satisfy

$$\sup_{(k,\theta,e)\in\{0,\dots,n\}\times\Delta_n\times E^n} ||Y^k(\theta_{(k)}, e_{(k)})||_{\mathcal{S}^\infty_{\mathbb{F}}} \quad < \quad \infty .$$

In the same way,

$$\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] = \int_{\Delta_n\times E^n} \mathbb{E}\Big[\int_0^{\theta_1\wedge T} |Z_s^0|^2 ds + \sum_{k=1}^n \int_{\theta_k\wedge T}^{\theta_{k+1}\wedge T} |Z_s^k(\theta_{(k)}, e_{(k)})|^2 ds\Big] \gamma(\theta, e) d\theta \, \eta(de) .$$

Thus, $Z \in L^2_{\mathbb{G}}(W)$ since the processes $(Z^k(\theta_{(k)}, e_{(k)}))_{k\in\{0,\dots,n\}}$ satisfy

$$\int_{\Delta_n\times E^n} \mathbb{E}\Big[\int_0^{\theta_1\wedge T} |Z_s^0|^2 ds + \sum_{k=1}^n \int_{\theta_k\wedge T}^{\theta_{k+1}\wedge T} |Z_s^k(\theta_{(k)}, e_{(k)})|^2 ds\Big] \gamma(\theta, e) d\theta \, \eta(de) \quad < \quad \infty .$$

□

We now give some explicit examples where the previous general theorem can be applied to provide existence of solution to BSDEs with jumps.

Corollary 3.3.1. *Suppose that the random variable ξ is bounded. Suppose also that the generator $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E \to \mathbb{R}$ satisfies one of those two conditions*

(i) f is deterministic and Lipschitz: there exists a constant C such that

$$|f(t, y, z, u(e) - y)_{e\in E} - f(t, y', z', u(e) - y')_{e\in E}| \quad \leq \quad C(|y - y'| + |z - z'|) ,$$

for all $(t, y, y', z, z', u) \in [0,T] \times [\mathbb{R}]^2 \times [\mathbb{R}^d]^2 \times \mathbb{R}^E$,

(ii) f is quadratic in z: there exists a constant C such that

$$|f(t, y, z, u)| \quad \leq \quad C(1 + |z|^2) ,$$

for all $(t, y, z, u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$.

Then, BSDE (3.3.1) *admits a solution in* $\mathcal{S}_{\mathbb{G}}^{\infty} \times L_{\mathbb{G}}^2(W) \times L^2(\mu)$.

Proof. **Step 1.** First notice that since ξ is a bounded random variable, we can choose $\xi^k(\tau_{(k)}, \zeta_{(k)})$ bounded for each $k = 1, \ldots, n$. Indeed, if C is a positive constant such that we have $|\xi| \leq C$, $\mathbb{P} - a.s.$, then we have

$$\xi \quad = \quad \tilde{\xi}^0 \mathbb{1}_{T \leq \tau_1} + \sum_{k=1}^{n} \tilde{\xi}^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) \mathbb{1}_{\tau_k < T \leq \tau_{k+1}},$$

with $\tilde{\xi}^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) = (\xi^k(\tau_1, \ldots, \tau_k, \zeta_1, \ldots, \zeta_k) \wedge C) \vee -C$, for each $k = 1, \ldots, n$.

Step 2. We then prove the existence in the two previous cases.

(i) Since f is Lipschitz and deterministic, it is possible to choose for each $k \in \{0, \ldots, n\}$ the function $f^k(., \theta_{(k)}, e_{(k)})$ Lipschitz continuous by taking $f^k(., \theta_{(k)}, e_{(k)}) = f$. Choosing ξ^k bounded as in Step 1, we get from El Karoui and Quenez [57], the existence of a solution for each Brownian BSDE. Applying Theorem 3.3.1, we get the existence of a solution to BSDE (3.3.1).

(ii) Since f is quadratic in z, it is possible to choose the functions $(f^k(., \theta_{(k)}, e_{(k)}))_{0 \leq k \leq n}$ quadratic in z. Indeed, if C is a positive constant such that $|f(t, y, z, u)| \leq C(1 + |z|^2)$, for all $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$, $\mathbb{P} - a.s.$ and f has the following decomposition

$$f(t, y, z, u) \quad = \quad f^0(t, y, z, u) \mathbb{1}_{t < \tau_1} + \sum_{k=1}^{n} f^k(t, y, z, u, \tau_{(k)}, \zeta_{(k)}) \mathbb{1}_{\tau_k \leq t < \tau_{k+1}},$$

then, f satisfies the same decomposition with $\tilde{f}^k$ instead of f^k where

$$\tilde{f}^k(t, y, z, u) = \left[f^k(t, y, z, u) \wedge (C(1 + |z|^2))\right] \vee (-C(1 + |z|^2)\,,$$

for all $(t, y, z, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$.

Choosing ξ^k bounded as in Step 1, we get from Kobylanski [83], the existence of a solution for each Brownian BSDE. Applying Theorem 3.3.1, we get the existence of a solution to BSDE (3.3.1). □

3.3.2 Application to the pricing of a European option in a complete market with default

In this example, we assume that there is a single random time τ representing the time occurrence of a shock in the prices. We denote by N the associated pure jump process:

$$N_t \quad = \quad \mathbb{1}_{\tau \leq t}\ , \quad 0 \leq t \leq T\ ,$$

and by M the compensated martingale associated with N, that we suppose being equal to

$$M_t \quad = \quad N_t - \int_0^t \lambda_s ds \ , \quad 0 \leq t \leq T \ .$$

We consider a financial market which consists of

– a non-risky asset S^0, whose strictly positive price process is defined by

$$dS_t^0 \quad = \quad r_t S_t^0 dt \ , \quad 0 \leq t \leq T \ , \quad S_0^0 = 1 \ ,$$

with $r_t \geq 0$, for all $t \in [0, T]$,

– two risky assets with respective price processes S^1 and S^2 defined by

$$dS_t^1 \quad = \quad S_{t^-}^1 (\mu_t dt + \sigma_t dW_t + \beta dM_t) \ , \quad 0 \leq t \leq T \ , \quad S_0^1 = s_0^1 \ ,$$

and

$$dS_t^2 \quad = \quad S_t^2 (\bar{\mu}_t dt + \bar{\sigma}_t dW_t) \ , \quad 0 \leq t \leq T \ , \quad S_0^2 = s_0^2 \ ,$$

with $\sigma_t > 0$ and $\bar{\sigma}_t > 0$ and $\beta > -1$ (to guarantee that the price process S^1 always remains strictly positive).

We assume that the coefficients r, μ, $\bar{\mu}$, σ and $\bar{\sigma}$ have the following forms

$$\begin{cases} r_t = r^0 \mathbb{1}_{t<\tau} + r^1(\tau) \mathbb{1}_{t \geq \tau} \ , \\ \mu_t = \mu^0 \mathbb{1}_{t<\tau} + \mu^1(\tau) \mathbb{1}_{t \geq \tau} \ , \\ \bar{\mu}_t = \bar{\mu}^0 \mathbb{1}_{t<\tau} + \bar{\mu}^1(\tau) \mathbb{1}_{t \geq \tau} \ , \\ \sigma_t = \sigma^0 \mathbb{1}_{t<\tau} + \sigma^1(\tau) \mathbb{1}_{t \geq \tau} \ , \\ \bar{\sigma}_t = \bar{\sigma}^0 \mathbb{1}_{t<\tau} + \bar{\sigma}^1(\tau) \mathbb{1}_{t \geq \tau} \ . \end{cases}$$

Assumption 3.3.1. *The following proportionality relation holds true*

$$\frac{r^1 - \mu^1}{\sigma^1} \quad = \quad \frac{r^1 - \bar{\mu}^1}{\bar{\sigma}^1} .$$

The aim of this subsection is to provide an explicit price for any bounded $\mathcal{G}_T$-measurable European option ξ, together with a replicating strategy $\pi = (\pi^0, \pi^1, \pi^2)$ (π_t^i corresponds to the number of shares S^i hold at time t), i.e. this market is complete for bounded contingent claims.

Let $\pi = (\pi^0, \pi^1, \pi^2)$ be a self-financing strategy. The wealth process Y associated with this strategy satisfied

$$Y_t \quad = \quad \pi_t^0 S_t^0 + \pi_t^1 S_t^1 + \pi_t^2 S_t^2 \ , \quad 0 \leq t \leq T \ . \tag{3.3.7}$$

Since π is a self financing strategy, we have

$$dY_t \;=\; \pi_t^0 dS_t^0 + \pi_t^1 dS_t^1 + \pi_t^2 dS_t^2\,, \quad 0 \le t \le T\,.$$

Combining this last equation with (3.3.7), we get

$$\begin{aligned} dY_t \;=\; & \big(r_t Y_t + (\mu_t - r_t)\pi_t^1 S_t^1 + (\bar\mu_t - r_t)\pi_t^2 S_t^2\big)dt \\ & + \big(\pi_t^1 \sigma_t S_{t^-}^1 + \pi_t^2 \bar\sigma_t S_t^2\big)dW_t + \pi_t^1 \beta S_{t^-}^1 dM_t, \quad 0 \le t \le T\,. \end{aligned} \tag{3.3.8}$$

Let us define the processes Z and U by

$$Z_t = \pi_t^1 \sigma_t S_{t^-}^1 + \pi_t^2 \bar\sigma_t S_t^2 \quad \text{and} \quad U_t = \pi_t^1 \beta S_{t^-}^1\,, \quad 0 \le t \le T\,. \tag{3.3.9}$$

Using Assumption 3.3.1, the dynamics (3.3.8) can then be written under the form

$$dY_t = \Big[r_t Y_t - \frac{r_t - \bar\mu_t}{\bar\sigma_t} Z_t - \Big(\frac{r_t - \mu_t}{\beta} + \lambda_t - \frac{\sigma_t(r_t - \bar\mu_t)}{\beta\bar\sigma_t}\Big)U_t\Big]dt + Z_t dW_t + U_t dN_t,\ 0 \le t \le T.$$

Therefore, the problem of valuing and hedging the contingent claim ξ consists in solving the following BSDE

$$\left\{ \begin{aligned} -dY_t \;=\; & \Big[\tfrac{r_t - \bar\mu_t}{\bar\sigma_t} Z_t + \Big(\tfrac{r_t - \mu_t}{\beta} + \lambda_t - \tfrac{\sigma_t(r_t - \bar\mu_t)}{\beta\bar\sigma_t}\Big)U_t - r_t Y_t\Big]dt \\ & - Z_t dW_t - U_t dN_t\,, \quad 0 \le t \le T\,, \\ Y_T \;=\; & \xi\,. \end{aligned} \right. \tag{3.3.10}$$

Using the previous subsection, one knows that it is possible to obtain a solution of this BSDE by solving two Brownian BSDEs. From the form of the coefficients r, μ, $\bar\mu$, σ and $\bar\sigma$, the two Brownian BSDEs associated to (3.3.10) are

$$\left\{ \begin{aligned} -dY_t^1(\theta) \;=\; & \Big[\tfrac{r^1(\theta) - \bar\mu^1(\theta)}{\bar\sigma^1(\theta)} Z_t^1(\theta) - r^1(\theta) Y_t^1(\theta)\Big]dt - Z_t^1(\theta) dW_t\,, \quad 0 \le t \le T\,, \\ Y_T^1(\theta) \;=\; & \xi^1(\theta)\,, \end{aligned} \right. \tag{3.3.11}$$

and

$$\left\{ \begin{aligned} -dY_t^0 \;=\; & \Big[\tfrac{r^0 - \bar\mu^0}{\bar\sigma^0} Z_t + \Big(\tfrac{r^0 - \mu^0}{\beta} + \lambda_t - \tfrac{\sigma^0(r^0 - \bar\mu^0)}{\beta\bar\sigma^0}\Big)(Y_t^1(t) - Y_t^0) - r^0 Y_t^0\Big]dt \\ & - Z_t dW_t\,, \qquad 0 \le t \le T\,, \\ Y_T^0 \;=\; & \xi^0\,. \end{aligned} \right. \tag{3.3.12}$$

Since these BSDEs are linear, we have an explicit solution for each BSDE. We get the following formula for $Y^1(\theta)$:

$$Y_t^1(\theta) \;=\; \frac{1}{\Gamma_t^1(\theta)} \mathbb{E}\Big[\xi^1(\theta)\Gamma_T^1(\theta)\Big|\mathcal{F}_t\Big]\,, \qquad 0 \le t \le T\,,$$

with $\Gamma^1(\theta)$ defined by

$$\Gamma_t^1(\theta) \;=\; \exp\Big(\frac{r^1(\theta) - \bar\mu^1(\theta)}{\bar\sigma^1(\theta)} W_t - \frac{1}{2}\Big|\frac{r^1(\theta) - \bar\mu^1(\theta)}{\bar\sigma^1(\theta)}\Big|^2 t - r^1(\theta) t\Big)\,, \qquad 0 \le t \le T\,.$$

For Y^0 we get :

$$Y_t^0 = \frac{1}{\Gamma_t^0}\mathbb{E}\Big[\xi^0\Gamma_T^0 + \int_t^T c_s\Gamma_s^0 ds\Big|\mathcal{F}_t\Big],$$

with Γ^0 defined by

$$\Gamma_t^0 = \exp\Big(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t a_s ds\Big),$$

where the parameters a, b and c are given by

$$\begin{cases} a_t = -r^0 - \Big(\dfrac{r^0-\mu^0}{\beta} + \lambda_t - \dfrac{\sigma^0(r^0-\bar{\mu}^0)}{\beta\bar{\sigma}^0}\Big), \\ b_t = \dfrac{r^0-\bar{\mu}^0}{\bar{\sigma}^0}, \\ c_t = \Big(\dfrac{r^0-\mu^0}{\beta} + \lambda_t - \dfrac{\sigma^0(r^0-\bar{\mu}^0)}{\beta\bar{\sigma}^0}\Big)Y_t^1(t). \end{cases}$$

The price at time t of the European option ξ is equal to Y_t^0 if $t<\tau$ and $Y_t^1(\tau)$ if $t\geq\tau$. Once we know the process Y and Z, a hedging strategy $\pi=(\pi^0,\pi^1,\pi^2)$ is given by (3.3.7) and (3.3.9).

3.3.3 Uniqueness

In this subsection, we provide a uniqueness result based on a comparison theorem. We consider two BSDEs with coefficients $(\underline{f},\underline{\xi})$ and $(\bar{f},\bar{\xi})$. We denote by $(\underline{Y},\underline{Z},\underline{U})$ and $(\bar{Y},\bar{Z},\bar{U})$ their respective solutions in $\mathcal{S}_{\mathbb{G}}^\infty\times L_{\mathbb{G}}^2(W)\times L^2(\mu)$. We consider the decomposition $(\underline{\xi}^k)_{0\leq k\leq n}$ (resp. $(\bar{\xi}^k)_{0\leq k\leq n}$, $(\underline{f}^k)_{0\leq k\leq n}$, $(\bar{f}^k)_{0\leq k\leq n}$, $(\underline{Y}^k)_{0\leq k\leq n}$, $(\bar{Y}^k)_{0\leq k\leq n}$, $(\underline{Z}^k)_{0\leq k\leq n}$, $(\bar{Z}^k)_{0\leq k\leq n}$, $(\underline{U}^k)_{0\leq k\leq n}$, $(\bar{U}^k)_{0\leq k\leq n}$) of $\underline{\xi}$ (resp. $\bar{\xi}$, $\underline{f}$, $\bar{f}$, $\underline{Y}$, $\bar{Y}$, $\underline{Z}$, $\bar{Z}$, $\underline{U}$, $\bar{U}$) (to alleviate notation, we shall omit the dependence on $(\theta_{(k)},e_{(k)})$). For ease of notation, we shall write:

- $\underline{F}^n(t,y,z,.)$ and $\bar{F}^n(t,y,z,.)$ instead of $\underline{f}^n(t,y,z,0,.)$ and $\bar{f}^n(t,y,z,0,.)$,
- $\underline{F}^k(t,y,z,.)$ and $\bar{F}^k(t,y,z,.)$ instead of $\underline{f}^k(t,y,z,\underline{Y}_t^{k+1}(t,.)-y,.)$ and $\bar{f}^k(t,y,z,\bar{Y}_t^{k+1}(t,.)-y,.)$ for each $k=0,\ldots,n-1$.

Before giving the comparison result, we first need an assumption on the behavior of the jumps w.r.t the filtration $\mathbb{G}$.

Assumption 3.3.2. *The stopping times $(\tau_k)_{1\leq k\leq n}$ are inaccessible in the filtration $\mathbb{G}$.*

We can state the general comparison theorem.

Theorem 3.3.2. *Suppose that $\underline{\xi}\leq\bar{\xi}$, $\mathbb{P}$-a.s. If for each $k=0,\ldots,n$,*

$$\underline{F}^k(t,y,z) \leq \bar{F}^k(t,y,z), \quad \forall(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d, \quad \mathbb{P}-a.s.,$$

and the generator $\bar{F}^k$ or $\underline{F}^k$ satisfy a comparison theorem for Brownian BSDEs. Then, under Assumption 3.3.2, if $\bar{U}_t = \underline{U}_t = 0$, $\forall\, t > \tau_n$, we have

$$\underline{Y}_t \quad \leq \quad \bar{Y}_t \ , \quad \forall\, t \in [0,T] \ , \quad \mathbb{P} - a.s.$$

Proof. **Step 1.** Notice that we can assume that $\underline{\xi}^k \leq \bar{\xi}^k$, $\mathbb{P} - a.s.$ Indeed, if it is not the case we can replace $\underline{\xi}^k$ (resp. $\bar{\xi}^k$) by $\underline{\xi}^k \wedge \bar{\xi}^k$ (resp. $\underline{\xi}^k \vee \bar{\xi}^k$).

Step 2. Since $(\bar{Y}, \bar{Z}, \bar{U})$ (resp. $(\underline{Y}, \underline{Z}, \underline{U})$) is solution to the BSDE with parameters $(\bar{\xi}, \bar{f})$ (resp. $(\underline{\xi}, \underline{f})$), we obtain from the decomposition in the filtration $\mathbb{F}$ that $(\bar{Y}^n, \bar{Z}^n)$ (resp. $(\underline{Y}^n, \underline{Z}^n)$) is solution to

$$\begin{aligned} \bar{Y}^n_t(\tau_{(n)}, \zeta_{(n)}) &= \bar{\xi}^n(\tau_n, \zeta_n) \\ &\quad + \int_t^T \bar{F}^n\Big(s, \bar{Y}^n_s(\tau_{(n)}, \zeta_{(n)}), \bar{Z}^n_s(\tau_{(n)}, \zeta_{(n)}), \tau_{(n)}, \zeta_{(n)}\Big) ds \\ &\quad - \int_t^T \bar{Z}^n_s(\tau_{(n)}, \zeta_{(n)}) dW_s \ , \quad \tau_n \leq t \leq T \ , \end{aligned} \tag{3.3.13}$$

$$\begin{aligned} \text{(resp.} \quad \underline{Y}^n_t(\tau_{(n)}, \zeta_{(n)}) &= \underline{\xi}^n(\tau_{(n)}, \zeta_{(n)}) \\ &\quad + \int_t^T \underline{F}^n\Big(s, \underline{Y}^n_s(\tau_{(n)}, \zeta_{(n)}), \underline{Z}^n_s(\tau_{(n)}, \zeta_{(n)}), \tau_{(n)}, \zeta_{(n)}\Big) ds \\ &\quad - \int_t^T \underline{Z}^n_s(\tau_{(n)}, \zeta_{(n)}) dW_s \ , \quad \tau_n \leq t \leq T \ ) \end{aligned} \tag{3.3.14}$$

and $(\bar{Y}^k, \bar{Z}^k)$ (resp. $(\underline{Y}^k, \underline{Z}^k)$) is solution to

$$\begin{aligned} \bar{Y}^k_t(\tau_{(k)}, \zeta_{(k)}) &= \bar{Y}^{k+1}_{\tau_{k+1}}(\tau_{(k+1)}, \zeta_{(k+1)}) - \bar{U}_{\tau_{k+1}}(\zeta_{k+1}) \\ &\quad + \int_t^{\tau_{k+1}} \bar{F}^k\Big(s, \bar{Y}^k_s(\tau_{(k)}, \zeta_{(k)}), \bar{Z}^k_s(\tau_{(k)}, \zeta_{(k)}), \tau_{(k)}, \zeta_{(k)}\Big) ds \\ &\quad - \int_t^{\tau_{k+1}} \bar{Z}^k_s(\tau_{(k)}, \zeta_{(k)}) dW_s \ , \quad \tau_k \leq t < \tau_{k+1} \ , \end{aligned} \tag{3.3.15}$$

$$\begin{aligned} \text{(resp.} \quad \underline{Y}^k_t(\tau_{(k)}, \zeta_{(k)}) &= \underline{Y}^{k+1}_{\tau_{k+1}}(\tau_{(k+1)}, \zeta_{(k+1)}) - U_{\tau_{k+1}}(\zeta_{k+1}) \\ &\quad + \int_t^{\tau_{k+1}} \underline{F}^k\Big(s, \underline{Y}^k_s(\tau_{(k)}, \zeta_{(k)}), \underline{Z}^k_s(\tau_{(k)}, \zeta_{(k)}), \tau_{(k)}, \zeta_{(k)}\Big) ds \\ &\quad - \int_t^{\tau_{k+1}} \underline{Z}^k_s(\tau_{(k)}, \zeta_{(k)}) dW_s \ , \quad \tau_k \leq t < \tau_{k+1} \ ) \end{aligned} \tag{3.3.16}$$

for each $k = 0, \ldots, n-1$.

Step 3. We introduce a family of processes $(\tilde{\bar{Y}}^k)_{0\leq k\leq n}$ (resp. $(\underline{\tilde{Y}}^k)_{0\leq k\leq n}$). We define it recursively by

$$\tilde{\bar{Y}}_t^n \;=\; \bar{Y}_t^n \mathbb{1}_{t\geq\tau_n} \text{ (resp. } \underline{\tilde{Y}}_t^n \;=\; \underline{Y}_t^n \mathbb{1}_{t\geq\tau_n}) \;, \quad 0\leq t\leq T \;,$$

and for $k=0,\ldots,n-1$

$$\begin{aligned} \tilde{\bar{Y}}_t^k &= \bar{Y}_t^k \mathbb{1}_{\tau_k\leq t<\tau_{k+1}} + \tilde{\bar{Y}}_t^{k+1}\mathbb{1}_{t\geq\tau_{k+1}} \\ \text{(resp. } \underline{\tilde{Y}}_t^k &= \underline{Y}_t^k \mathbb{1}_{\tau_k\leq t<\tau_{k+1}} + \underline{\tilde{Y}}_t^{k+1}\mathbb{1}_{t\geq\tau_{k+1}}) \;, \quad 0\leq t\leq T \;. \end{aligned}$$

These processes are càd-làg with jumps only at times τ_l, $l=1,\ldots,n$. Notice also that $\tilde{\bar{Y}}^n$ (resp. $\underline{\tilde{Y}}^n$, $\tilde{\bar{Y}}^k$, $\underline{\tilde{Y}}^k$) satisfies equation (3.3.13) (resp. (3.3.14), (3.3.15), (3.3.16)).

Step 4. We prove by a backward induction that $\underline{\tilde{Y}}^n \leq \tilde{\bar{Y}}^n$ on $[\tau_n,T]$ and $\underline{\tilde{Y}}^k \leq \tilde{\bar{Y}}^k$ on $[\tau_k,\tau_{k+1})$, for each $k=0,\ldots,n-1$.

• Since $\underline{\xi}^n \leq \bar{\xi}^n$ and $\bar{F}^n$ or $\underline{F}^n$ satisfy a comparison theorem for Brownian BSDEs we immediately get from (3.3.13) and (3.3.14)

$$\underline{Y}_t^n(\tau_{(n)},\zeta_{(n)}) \;\leq\; \bar{Y}_t^n(\tau_{(n)},\zeta_{(n)}) \;, \quad \tau_n\leq t\leq T \;.$$

• Fix $k\leq n-1$ and suppose that $\underline{\tilde{Y}}_t^{k+1} \leq \tilde{\bar{Y}}_t^{k+1}$, $\forall\, t\in[\tau_{k+1},\tau_{k+2})$. Denote by ${}^p\tilde{\bar{Y}}^l$ (resp. ${}^p\underline{\tilde{Y}}^l$) the predictable projection of $\tilde{\bar{Y}}^l$ (resp. $\underline{\tilde{Y}}^l$) for each $l=0,\ldots,n$. Since $\tilde{\bar{Y}}^l$ (resp. $\underline{\tilde{Y}}^l$) has inaccessible jumps, we have

$${}^p\tilde{\bar{Y}}_t^l \;=\; \tilde{\bar{Y}}_{t^-}^l \quad \text{(resp. } {}^p\underline{\tilde{Y}}_t^l \;=\; \underline{\tilde{Y}}_{t-}^l) \;, \quad 0\leq t\leq T \;.$$

From equations (3.3.15) and (3.3.16), and the definition of $\tilde{\bar{Y}}^l$ (resp. $\underline{\tilde{Y}}^l$), we have for $l=k$

$$\begin{aligned} {}^p\tilde{\bar{Y}}_t^k &= {}^p\tilde{\bar{Y}}_{\tau_{k+1}}^{k+1} + \int_t^{\tau_{k+1}} \bar{F}^k\Big(s,{}^p\tilde{\bar{Y}}_s^k,\bar{Z}_s^k,\tau_{(k)},\zeta_{(k)}\Big)ds \\ &\quad -\int_t^{\tau_{k+1}} \bar{Z}_s^k dW_s \;, \quad \tau_k\leq t<\tau_{k+1} \;, \end{aligned} \tag{3.3.17}$$

$$\begin{aligned} \text{(resp. } {}^p\underline{\tilde{Y}}_t^k &= {}^p\underline{\tilde{Y}}_{\tau_{k+1}}^{k+1} + \int_t^{\tau_{k+1}} \underline{F}^k\Big(s,{}^p\underline{\tilde{Y}}_s^k,\underline{Z}_s^k,\tau_{(k)},\zeta_{(k)}\Big)ds \\ &\quad -\int_t^{\tau_{k+1}} \underline{Z}_s^k dW_s \;, \quad \tau_k\leq t<\tau_{k+1} \;). \end{aligned} \tag{3.3.18}$$

Since $\tilde{\bar{Y}}_{\tau_{k+1}}^{k+1} \geq \underline{\tilde{Y}}_{\tau_{k+1}}^{k+1}$, we get ${}^p\tilde{\bar{Y}}_{\tau_{k+1}}^{k+1} \geq {}^p\underline{\tilde{Y}}_{\tau_{k+1}}^{k+1}$. This together with conditions on $\bar{F}^k$ and $\underline{F}^k$ give the result.

Step 5. Since $\tilde{\bar{Y}}^k$ (resp. $\underline{\tilde{Y}}^k$) coincides with $\bar{Y}$ (resp. $\underline{Y}$) on $[\tau_k,\tau_{k+1})$, we get the result. □

In this form, the previous theorem is not usable since the condition on the generators of the Brownian BSDEs is implicit: it involves the solution to the previous Brownian BSDE at each step. We give in the sequel, an explicit example for the case of quadratic generators. We use the result on quadratic BSDEs obtained in Kobylanski [83]. We introduce an assumption similar to the one detailed in Kobylanski [83].

Assumption 3.3.3. *There exists a constant C such that*

$$\begin{cases} |f(t,y,z,u)| & \leq & C(1+|z|^2)\;, \\ \Big|\partial_z f(t,y,z,u)\Big| & \leq & C(1+|z|)\;, \end{cases} \tag{3.3.19}$$

for all $(t,y,z,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$, $\mathbb{P}$-a.s. For all $\varepsilon > 0$, there exists a constant C_ε such that

$$\partial_y f(t,y,z,(u(e)-y)_{e\in E}) \quad \leq \quad C_\varepsilon + \epsilon|z|^2\;,$$

for all $(t,y,z,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^E$, $\mathbb{P}$-a.s.

We can now state our uniqueness result for quadratic BSDEs with jumps.

Theorem 3.3.3. *Under Assumption 3.3.3, BSDE* (3.3.1) *admits at most one solution.*

Proof. The proof is a consequence of Theorem 3.3.2 and Theorem 2.6 in [83]. □

3.4 Decomposition of Feynman-Kac formula for Integral Partial Differential Equation (IPDE)

In this section, we aim at giving a new Feynman-Kac formula for IPDE, which could be regarded as a decomposition of Feynman-Kac formula given by Barles, Buckdahn and Pardoux [7]. We denote to simplify $\tilde{h}(x,u(t,x),\sigma Du(t,x),(u(t,x+\beta(x,e)))_{e\in E})$ instead of $h(x,u(t,x),\sigma Du(t,x),\int_E(u(t,x+\beta(x,e))-u(t,x))\gamma(x,e)\lambda(de))$. And we consider the following IPDE

$$\begin{cases} -\partial_t u(t,x) - \mathcal{L}u(t,x) - \tilde{h}(x,u(t,x),\sigma Du(t,x),(u(t,x+\beta(x,e)))_{e\in E}) = 0 \\ \text{for } (t,x) \in [0,T] \times \mathbb{R}^d \text{ and} \\ u(T,.) = g(.), \end{cases} \tag{3.4.1}$$

here, $\mathcal{L}$ is a local second order operator given by [3]

$$\mathcal{L}u(t,x) \quad = \quad b(x)Du(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(x)D^2u(t,x)).$$

We make the following assumptions:

[3] Let A be a square matrix, then $A^\top$ stands for the transposition of the matrix A and Tr(A) is the sum of the elements on the main diagonal of the matrix A.

Assumption 3.4.1.
- $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and has a linear growth: $|b(x)| \leq c(1+|x|)\ \forall x \in \mathbb{R}^d$,
- $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is Lipschitz continuous and has a linear growth: $|\sigma(x)| \leq c(1+|x|)\ \forall x \in \mathbb{R}^d$,
- $g : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous and has a linear growth: $|g(x)| \leq c(1+|x|)\ \forall x \in \mathbb{R}^d$,
- $h : \mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and has a linear growth: $|h(x,0,0,0)| \leq c(1+|x|)\ \forall x \in \mathbb{R}^d$, and $p \mapsto h(x,y,z,p)$ is nondecreasing for all $(x,y,z) \in \mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$,
- the function γ (resp. β) : $\mathbb{R}^d \times E \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(E)$-measurable bounded and Lipschitz w.r.t x uniformly in $e \in E$:

$$|\gamma(x,e) - \gamma(x',e)| \leq c|x-x'|\ \forall\,(x,x',e) \in \mathbb{R}^d\times\mathbb{R}^d\times E,$$

$$(resp.\ |\beta(x,e) - \beta(x',e)| \leq c|x-x'|\ \forall\,(x,x',e) \in \mathbb{R}^d\times\mathbb{R}^d\times E),$$

- $\lambda : E \to \mathbb{R}$ is $\mathcal{B}(E)$-measurable and nonnegative.

We also assume that the Poisson random measure μ is independent of W and that it admits the intensity λ. For any predictable λ-square integrable process $U : \Omega\times[0,T]\times E \to \mathbb{R}$, we have

$$\mathbb{E}\Big[\int_0^T\int_E U_s(e)\mu(de,ds)\Big] \;=\; \mathbb{E}\Big[\int_0^T\int_E U_s(e)\lambda(de)ds\Big].$$

Under these assumptions, we know (see [7]) that $Y_t^{t,x} = v(t,x)$ where v is the unique solution to (3.4.1) and $(Y^{t,x}, Z^{t,x}, U^{t,x}) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}(W)\times L^2(\mu)$ is the solution on $[t,T]$ to the BSDE

$$\begin{aligned} Y_s &= g(X_T^{t,x}) + \int_s^T h\Big(X_r^{t,x}, Y_r, Z_r, \int_E U_r(e)\gamma(X_r^{t,x},e)\lambda(de)\Big)dr \\ &\quad - \int_s^T Z_r dW_r - \int_s^T\int_E U_r(e)\mu(de,dr)\,, \quad \text{for } t \leq s \leq T, \end{aligned}$$

and $X^{t,x}$ is the jump diffusion defined by

$$\begin{cases} dX_s^{t,x} = b(X_{s^-}^{t,x})ds + \sigma(X_{s^-}^{t,x})dW_s + \displaystyle\int_E \beta(X_{s^-}^{t,x},e)\mu(de,ds)\,, \quad \text{for } t \leq s \leq T, \\ X_t^{t,x} = x. \end{cases}$$

We introduce the decomposition of the diffusion X as in Lemma 3.2.1

$$X_s^{t,x} \;=\; X_s^{0,t,x}\mathbb{1}_{s\leq\tau_1} + \sum_{k=1}^{n-1} X_s^{k,t,x}(\tau_{(k)},\zeta_{(k)})\mathbb{1}_{\tau_k\leq s<\tau_{k+1}} + X_s^{n,t,x}(\tau,\zeta)\mathbb{1}_{\tau_n\leq s}\,,$$

where $(X_s^{k,t,x})_{t\leq s\leq T}$ is defined by

$$\begin{aligned} X_s^{k,t,x}(\theta_{(k)}, e_{(k)}) &= x + \int_t^s b\big(X_r^{k,t,x}(\theta_{(k)}, e_{(k)})\big)dr + \int_t^s \sigma\big(X_r^{k,t,x}(\theta_{(k)}, e_{(k)})\big)dW_r \\ &\quad + \sum_{i=1}^k \mathbb{1}_{t\leq\theta_i\leq s}\beta\big(X_{\theta_i^-}^{k,t,x}(\theta_{(k)}, e_{(k)}), e_i\big) \end{aligned}$$

with the convention $X_{t^-}^{k,t,x}(\theta_{(k)}, e_{(k)}) = x$.
Then, we have by Theorem 3.3.1 that

$$Y_s^{t,x} = Y_s^{0,t,x}\mathbb{1}_{s<\tau_1} + \sum_{k=1}^{n-1} Y_s^{k,t,x}(\tau_{(k)}, \zeta_{(k)})\mathbb{1}_{\tau_k\leq s<\tau_{k+1}} + Y_s^{n,t,x}(\tau,\zeta)\mathbb{1}_{\tau_n\leq s}\,,$$

where $(Y^{n,t,x}(\theta,e), Z^{n,t,x}(\theta,e)) \in \mathcal{S}_{\mathbb{F}}^{\infty}\times L_{\mathbb{F}}^2(W)$ is the solution to

$$Y_s = g(X_T^{n,t,x}) + \int_s^T h(X_r^{n,t,x}, Y_r, Z_r, 0)dr - \int_s^T Z_r dW_r\,,$$

and for each $k = 0,\ldots,n-1$, $(Y^{k,t,x}(\theta_{(k)}, e_{(k)}), Z^{k,t,x}(\theta_{(k)}, e_{(k)})) \in \mathcal{S}_{\mathbb{F}}^{\infty}\times L_{\mathbb{F}}^2(W)$ is solution to

$$Y_s = g(X_T^{k,t,x}) + \int_s^T \tilde{h}(X_r^{k,t,x}, Y_r, Z_r, (Y_r^{k+1,t,x}(r,e))_{e\in E})ds - \int_s^T Z_r dW_r.$$

Since μ is a Poisson measure, the probability for that two defaults appear simultaneously is equal to 0. Thus, it is possible to restrict the set Δ to its interior to define the processes $Y^k(\tau_{(k)}, \zeta_{(k)})$.

Using the link between Brownian BSDEs and parabolic second order PDEs (see Peng [108]), we have

$$Y_s^{n,t,x}(\theta,e) = v_n(s, X_s^{n,t,x}, \theta, e) \ \text{ for } s\geq\theta_n\,,$$

where $v_n(.,\theta,e)$ is the solution to

$$-\partial_t v_n(.,\theta,e) - \mathcal{L}v_n(.,\theta,e) - h(., v_n(.,\theta,e), \sigma D v_n(.,\theta,e), 0) = 0\,, \tag{3.4.2}$$

with terminal condition

$$v_n(T,x,\theta,e) = g(x)\mathbb{1}_{\theta_n<T}\,. \tag{3.4.3}$$

And for each $k = 0,\ldots,n-1$, we have

$$Y_s^{k,t,x} = v_k(s, X_s^{k,t,x}, \theta_{(k)}, e_{(k)}) \ \text{ for } s\geq\theta_k\,,$$

where $v_k(.,\theta_{(k)}, e_{(k)})$ is the solution to

$$\begin{aligned} -\partial_t v_k(.,\theta_{(k)}, e_{(k)}) - \mathcal{L}v_k(.,\theta_{(k)}, e_{(k)}) - \tilde{h}(v_k(.,\theta_{(k)}, e_{(k)}), \sigma D v_k(.,\theta_{(k)}, e_{(k)}), \\ (v_{k+1}(t, x+\beta(x,e), \theta_{(k)}, t, e_{(k)}, e))_{e\in E}) = 0\,, \end{aligned} \tag{3.4.4}$$

with terminal condition

$$v_k(T, x, \theta_{(k)}, e_{(k)}) \quad = \quad g(x)\mathbb{1}_{\theta_k < T} \,. \tag{3.4.5}$$

Indeed, using the identification between $Y^{n,t,x}$ and v_n, we have that $(Y^{n-1,t,x}, Z^{n-1,t,x})$ is solution to

$$Y_s = g(X_T^{n-1,t,x}) + \int_s^T \tilde{h}(X_r^{n-1,t,x}, Y_r, Z_r, (Y_r^{n,t,x}(r,e))_{e \in E})dr - \int_s^T Z_r dW_r \,, \tag{3.4.6}$$

for $\theta_{n-1} \leq s \leq T$. From the definition of $X^{n,t,x}$, we have

$$X_r^{n,t,x}(r,e) = X_r^{n-1,t,x} + \beta(X_r^{n-1,t,x}, e) \,. \tag{3.4.7}$$

(3.4.6) and (3.4.7) give that $(Y^{n-1,t,x}, Z^{n-1,t,x})$ is solution to

$$\begin{aligned} Y_s \quad = \quad & g(X_T^{n-1,t,x}) + \int_s^T \tilde{h}(X_r^{n-1,t,x}, Y_r, Z_r, (v_n(r, X_r^{n-1,t,x}, r, e) + \beta(X_r^{n-1,t,x}, e))_{e \in E})dr \\ & - \int_t^T Z_s dW_s, \end{aligned}$$

for $\theta_{n-1} \leq s \leq T$. Using the link between Brownian BSDEs and PDEs, we obtain that

$$Y_s^{n-1,t,x} \quad = \quad v_{n-1}(s, X_s^{n-1,t,x}) \ \text{ for } s \geq \theta_{n-1} \,,$$

where $v_{n-1}(.)$ is solution to (3.4.4)-(3.4.5). Iterating this argument until $k = 0$, we get the result. Finally, we have the following theorem.

Theorem 3.4.1. *Let v be the unique solution to (3.4.1). Then, we have*

$$v(t,x) \quad = \quad Y_t^{0,t,x}.$$

Moreover, we have the following "decomposition" of the function v

$$v(t,x) \quad = \quad v_0(t,x) \,,$$

where the family $(v_k(., \theta_{(k)}, e_{(k)}))_{0 \leq k \leq n, \, \theta_{(n)} \in \Delta_n, \, e_{(n)} \in E^n}$ is defined by the terminal PDE (3.4.2)-(3.4.3) and the recursive system of PDEs (3.4.4)-(3.4.5) for each $k = 0, \ldots, n$.

3.5 Utility maximization in a jump market model

One of the important problems in mathematical finance is the valuation of contingent claims in incomplete financial markets. We consider a financial market model with a riskless bond assumed for simplicity equal to one, and one risky asset subjects to some counterparty risks.

The dynamics of the risky asset is affected by other firms, the counterparties, which may default at some random times, inducing consequently some jumps in the asset price. However, this stock still exists and can be traded after the default of the counterparties. We keep the notation of previous sections.

In the sequel, we shall make the standing assumption, called **(H)**-hypothesis, on the enlarged progressive filtration: any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

We consider that the price process $(S_t)_{0\leq t\leq T}$ evolves according to the equation

$$dS_t = S_{t^-}\Big(\mu_t dt + \sigma_t dW_t + \int_E \beta_t(e)\mu(de,dt)\Big) .$$

All processes μ, σ and β are assumed to be $\mathbb{G}$-predictable and uniformly bounded. Moreover, we assume that the process $(\sigma_t)_{0\leq t\leq T}$ is positive, and the process $(\beta_t)_{0\leq t\leq T}$ satisfies $\beta_{\tau_i}(e) > -1$ for each $i = 1,\ldots,n$ and any $e \in E$. This last condition implies that the process S is almost surely positive. We also suppose that $\theta_t = \sigma_t^{-1}\mu_t$ is bounded.
A self-financing trading strategy is determined by its initial capital $x \in \mathbb{R}$ and the amount of money π_t invested in the stock, at time $t \in [0,T]$. Formally, π/S is in the space $L(S)$ of $\mathbb{G}$-predictable S-integrable $\mathbb{R}$-valued processes so that the stochastic integral $\int_0^t \frac{\pi_s}{S_s} dS_s$ is well defined. The wealth at time t associated with a strategy (x,π) is

$$X_t^{x,\pi} \quad = \quad x + \int_0^t \pi_s\mu_s ds + \int_0^t \pi_s\sigma_s dW_s + \int_0^t\int_E \pi_s\beta_s(e)\mu(de,ds), \quad t \in [0,T].$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a utility function and B a contingent claim, that is a random payoff at time T described by the $\mathcal{G}_T$-measurable random variable B. We suppose that B is bounded. Then, we define

$$V^B(x) \quad = \quad \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[G\big(X_T^{x,\pi} - B\big)\big], \tag{3.5.1}$$

the maximal expected utility we can achieve by starting at time 0 with initial capital x, using some strategy $\pi \in \mathcal{A}$ (which is defined in the sequel) on $[0,T]$ and paying B at time T.
In the sequel we use the utility indifference approach to define the price at the initial time of the contingent claim. We define the price $C(x,B)$ implicitly by the requirement that

$$C(x,B) \quad = \quad \inf\Big\{p \in \mathbb{R} \ : \ V^0(x) \ = \ V^B(x+p)\Big\}.$$

In terms of expected utility, the indifference price for B is the amount of initial capital such that the investor is indifferent between holding or not the contingent claim.

To pass from the above formal definitions to rigorous results, we now choose one particular utility function G. Throughout the rest of this paper, we work with the exponential utility function

$$G(x) \quad = \quad -\exp(-\alpha x)\,,$$

where α is a given positive constant which can be seen as a coefficient of absolute risk aversion.

Finally, we define the space $\mathcal{A}$ of admissible strategies.

Definition 3.5.1. Let C be a compact set in $\mathbb{R}$. A predictable $\mathbb{R}$-valued process π is an admissible trading strategy, if it takes its values in C, i.e. $\pi_t \in C,\ dt \otimes d\mathbb{P} - a.s.$

This case is studied by Morlais [99]. We give here another proof for the existence of a solution.

The set consisting of all constrained strategies satisfies an additional integrability property:

Lemma 3.5.1. *All trading strategies $\pi = (\pi_t)_{0\leq t\leq T}$ as introduced in Definition 3.5.1 satisfy*

$$\{\exp(-\alpha X_\tau^{x,\pi}),\quad \tau\ \mathcal{G}-\textit{stopping time with values in } [0,T]\}\ \textit{is a uniformly integrable family.}$$

Proof. We consider the process $L_t = \exp(-\alpha X_t^{x,\pi})$ for any $\pi \in \mathcal{A}$. From Itô's formula, we get

$$dL_t \;=\; L_{t^-}\Big[-\alpha\pi_t\sigma_t dW_t + \int_E \big(e^{-\alpha\pi_s\beta_s(e)} - 1\big)\mu(de,dt)\Big] + L_t\Big[\frac{\alpha^2}{2}|\pi_t\sigma_t|^2 - \alpha\pi_t\mu_t\Big]dt.$$

Using Doleans-Dade exponential, we get

$$L_t \;=\; L_0\mathcal{E}\Big(\int_0^t -\alpha\pi_s\sigma_s dW_s + \int_0^t\int_E \big(e^{-\alpha\pi_s\beta_s(e)} - 1\big)\tilde{\mu}(de,ds)\Big)e^{A_t^\pi},$$

with

$$A_t^\pi \;=\; \int_0^t \Big[\frac{\alpha^2}{2}|\pi_s\sigma_s|^2 - \alpha\pi_s\mu_s + \int_E \big(e^{-\alpha\pi_s\beta_s(e)} - 1\big)\lambda(de)\Big]ds.$$

This process A^π is bounded with the assumptions on the coefficients and the definition of $\mathcal{A}$. Moreover, using Kamazaki's criterion the stochastic exponential in the process L is a true martingale. Hence, we get the uniform integrability condition. □

In order to characterize the value function $V^B(x)$ and an optimal strategy, we construct, as in Hu *et al.* [67] and Morlais [99], a family of stochastic processes $R^{(\pi)}$ with the following properties:

(i) $R_T^{(\pi)} = -\exp(-\alpha(X_T^{x,\pi} - B))$ for all $\pi \in \mathcal{A}$,

(ii) $R_0^{(\pi)} = R_0$ is constant for all $\pi \in \mathcal{A}$,

(iii) $R^{(\pi)}$ is a supermartingale for all $\pi \in \mathcal{A}$ and there exists a $\hat{\pi} \in \mathcal{A}$ such that $R^{(\hat{\pi})}$ is a martingale.

Given processes possessing these properties we can compare the expected utilities of the strategies $\pi \in \mathcal{A}$ and $\hat{\pi} \in \mathcal{A}$ by

$$\mathbb{E}\big[-\exp\big(-\alpha\big(X_T^{x,\pi} - B\big)\big)\big] \;\leq\; R_0(x) \;=\; \mathbb{E}\big[-\exp\big(-\alpha\big(X_T^{x,\hat{\pi}} - B\big)\big)\big] \;=\; V^B(x),$$

whence $\hat{\pi}$ is the desired optimal strategy. To construct this family, we set

$$R_t^{(\pi)} \;=\; -\exp\big(-\alpha\big(X_t^{x,\pi} - Y_t\big)\big), \;\; t \in [0,T], \; \pi \in \mathcal{A},$$

where (Y, Z, U) is a solution of the BSDE

$$Y_t = B + \int_t^T f(s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds), \; 0 \leq t \leq T \, . \quad (3.5.2)$$

We have to choose a function f for which $R^{(\pi)}$ is a supermartingale for all $\pi \in \mathcal{A}$ and there exists a $\hat{\pi} \in \mathcal{A}$ such that $R^{(\hat{\pi})}$ is a martingale. We assume that there exists a triple (Y, Z, U) solving a BSDE with jumps of the form (3.5.2), with terminal condition B and with a driver f to be determined. We first apply a generalized Itô's formula to $R^{(\pi)}$ for any strategy π

$$\begin{aligned} dR_t^{(\pi)} \;&=\; R_{t^-}^{(\pi)}\Big[\Big(-\alpha(f(t, Z_t, U_t) + \pi_t\mu_t) + \frac{\alpha^2}{2}(\pi_t\sigma_t - Z_t)^2\Big)dt - \alpha(\pi_t\sigma_t - Z_t)dW_t \\ &\quad + \int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\mu(de, dt)\Big]. \end{aligned}$$

$R^{(\pi)}$ satisfies: $dZ_t = Z_{t^-}dM_t^{(\pi)} + Z_t dA_t^{(\pi)}$, with $A^{(\pi)}$ such that

$$\begin{cases} dM_t^{(\pi)} = \; -\alpha(\pi_t\sigma_t - Z_t)dW_t + \displaystyle\int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\tilde{\mu}(de, dt)\, , \\ dA_t^{(\pi)} = \Big(-\alpha(f(t, Z_t, U_t) + \pi_t\mu_t) + \dfrac{\alpha^2}{2}(\pi_t\sigma_t - Z_t)^2 \\ \qquad\quad + \displaystyle\int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\lambda(de)\Big)dt. \end{cases}$$

It follows that $R^{(\pi)}$ has the multiplicative form

$$R_t^{(\pi)} = R_0^{(\pi)}\mathcal{E}(M_t^{(\pi)})\exp(A_t^{(\pi)}),$$

where $\mathcal{E}(M^{(\pi)})$ denotes the Doleans-Dade exponential of the local martingale $M^{(\pi)}$. Since $\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1 \geq -1$, $\mathbb{P} - a.s.$, the Doleans-Dade exponential of the discontinuous part of $M^{(\pi)}$ is a positive local martingale and hence, a supermartingale. The supermartingale condition in (iii) holds true, provided, for all π, the process $\exp(A^{(\pi)})$ is nondecreasing, this entails

$$-\alpha(f(t, Z_t, U_t) + \pi_t\mu_t) + \frac{\alpha^2}{2}(\pi_t\sigma_t - Z_t)^2 + \int_E (\exp(-\alpha(\pi_t\beta_t(e) - U_t(e))) - 1)\lambda(de) \geq 0 \, .$$

This condition holds true, if we define f as follows

$$\begin{aligned} f(t,z,u) &= \inf_{\pi\in C}\Big\{\frac{\alpha}{2}\Big|\pi\sigma_t-\Big(z+\frac{\theta_t}{\alpha}\Big)\Big|^2+\int_E\frac{\exp(\alpha(u(e)-\pi\beta_t(e)))-1}{\alpha}\lambda(de)\Big\} \\ &\quad -\theta_t z-\frac{|\theta_t|^2}{2\alpha}\,, \end{aligned}$$

recall that $\theta_t=\mu_t/\sigma_t$ for $t\in[0,T]$.

Theorem 3.5.1. *The value function of the optimization problem (3.5.1) is given by*

$$V^B(x) \quad = \quad -\exp(-\alpha(x-Y_0)), \tag{3.5.3}$$

where Y_0 is defined as the initial value of the solution $(Y,Z,U)\in\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}(W)\times L^2(\mu)$ of the BSDE

$$Y_t=B+\int_t^T f(s,Z_s,U_s)ds-\int_t^T Z_sdW_s-\int_t^T\int_E U_s(e)\mu(de,ds),\ 0\le t\le T\ , \tag{3.5.4}$$

with

$$\begin{aligned} f(t,z,u) &= \inf_{\pi\in C}\Big\{\frac{\alpha}{2}\Big|\pi_t\sigma_t-\Big(z+\frac{\theta_t}{\alpha}\Big)\Big|^2+\int_E\frac{\exp(\alpha(u(e)-\pi_t\beta_t(e)))-1}{\alpha}\lambda(de)\Big\} \\ &\quad -\theta_t z-\frac{|\theta_t|^2}{2\alpha}\,. \end{aligned}$$

A trading strategy $\hat\pi\in\mathcal{A}$ is optimal if

$$\hat\pi_t\in\ \ \arg\min_{\pi\in C}\Big\{\tfrac{\alpha}{2}\Big|\pi_t\sigma_t-\Big(z+\tfrac{\theta_t}{\alpha}\Big)\Big|^2+\int_E\tfrac{\exp(\alpha(u(e)-\pi_t\beta_t(e)))-1}{\alpha}\lambda(de)\Big\}-\theta_t z-\tfrac{|\theta_t|^2}{2\alpha}\ ,$$

for all $t\in[0,T]$.

Remark 3.5.1. Note that the logarithm of the value function between two successive defaults is characterized as the solution of a Brownian BSDE.

Proof. **Step 1.** We first prove the existence of a solution to BSDE (3.5.4).

For that we apply Theorem 3.3.1. Let σ^k, θ^k and β^k, $k=0,\dots,n$, be the respective terms appearing in the decomposition of σ, θ and β given by Lemma 3.2.1.

Then, in the decomposition of the generator f, we can choose the functions f^k as

$$\begin{aligned} f^k(t,z,u) &= \inf_{\pi\in C}\Big\{\frac{\alpha}{2}\Big|\pi_t\sigma^k_t-\Big(z+\frac{\theta^k_t}{\alpha}\Big)\Big|^2+\int_E\frac{\exp(\alpha(u(e)-\pi_t\beta^k_t(e)))-1}{\alpha}\lambda(de)\Big\} \\ &\quad -\theta^k_t z-\frac{|\theta^k_t|^2}{2\alpha}\,. \end{aligned}$$

We first prove that the following BSDE admits for all $(\theta, e) \in \Delta_n \times E^n$ a solution $(Y^n(\theta,e), Z^n(\theta,e)) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}(W)$ (we shall omit the dependence on (θ, e))

$$Y_t^n = B^n + \int_t^T f^n(s, Z_s^n, 0)ds - \int_t^T Z_s^n dW_s\,, \quad 0 \leq t \leq T\,. \tag{3.5.5}$$

Since $0 \in C$, we have

$$-\theta_t^n z - \int_E \frac{\lambda(de)}{\alpha} - \frac{|\theta_t^n|^2}{2\alpha} \leq f^n(t,z,0) \leq \frac{\alpha}{2}|z|^2.$$

Therefore, we can apply Theorem 2.3 of [83], and we get that for any $(\theta, e) \in \Delta_n \times E^n$, there exists a solution to BSDE (3.5.5). From Proposition 2.1 of [83], we get the existence of a constant K such that

$$\sup_{(\theta,e)\in\Delta_n\times E^n} ||Y^n(\theta,e)||_{\mathcal{S}^\infty_{\mathbb{F}}} + \mathbb{E}\Big[\int_0^T |Z_t^n(\theta,e)|^2 dt\Big] \leq K\,.$$

We now prove by iteration for $k = 0, \ldots, n-1$, that the BSDE

$$Y_t^k = B^k + \int_t^T f^k(s, Z_s^k, Y_s^{k+1}(s,.) - Y_s^k)ds - \int_t^T Z_s^k dW_s\,, \quad 0 \leq t \leq T\,,$$

admits a solution $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)})) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}(W)$. We denote g^k the function defined by

$$g^k(t,y,z) = f^k(t, z, Y_t^{k+1}(t,.) - y)\,, \quad (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d\,.$$

Since C is compact, the function $g^k(t,.,.)$ is continuous. We also remark that g^k is nonincreasing in y. Hence, it is monotonic in y in the sense of [29]: there exists a constant M such that

$$(g^k(t,y,z) - g^k(t,y',z))(y-y') \leq M|y-y'|^2\,,$$

for all $(t, y, y', z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.

Moreover, since $0 \in C$, and $\sup_{(\theta_{(k+1)}, e_{(k+1)})} ||Y^{k+1}(\theta_{(k+1)}, e_{(k+1)})||_{\mathcal{S}^\infty_{\mathbb{F}}} < \infty$, we get the existence of a constant K such that

$$-\int_E \frac{\lambda(de)}{\alpha} - \theta_t^k z - \frac{|\theta_t^k|^2}{2\alpha} \leq g^k(t,y,z) \leq \frac{\alpha}{2}|z|^2 + \int_E \frac{\exp(\alpha(K-y))}{\alpha}\lambda(de)\,.$$

Since θ is bounded, we get the existence of a constant K (eventually different from the previous one) such that

$$|g^k(t,y,z)| \leq K(1 + |z|^2 + e^{\alpha|y|})\,.$$

We can then apply Theorem 2.1 of [29], and we obtain that the BSDE admits a solution $(Y^k(\theta_{(k)}, e_{(k)}), Z^k(\theta_{(k)}, e_{(k)})) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}(W)$ and that $\sup_{(\theta_{(k)}, e_{(k)})} ||Y^k(\theta_{(k)}, e_{(k)})||_{\mathcal{S}^\infty_{\mathbb{F}}} < \infty$.

Step 2. We now prove the uniqueness of a solution to BSDE (3.5.4). Let (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) be two solutions of (3.5.4) in $\mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}(W) \times L^2(\mu)$. Since Y^1 and Y^2 are bounded, these two triples are also solutions to

$$Y_t = B + \int_t^T \tilde{f}(s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\mu(de, ds) , \quad 0 \le t \le T , \tag{3.5.6}$$

with

$$\begin{aligned} \tilde{f}(t, z, u) &= \inf_{\pi \in C} \Big\{ \frac{\alpha}{2} \Big| \pi_t \sigma_t - \Big(z + \frac{\theta_t}{\alpha} \Big) \Big|^2 + \int_E \frac{\exp(\alpha(u(e) \wedge M - \pi_t \beta_t(e))) - 1}{\alpha} \lambda(de) \Big\} \\ &\quad - \theta_t z - \frac{|\theta_t|^2}{2\alpha} . \end{aligned}$$

Indeed, since the processes Y^1 and Y^2 are bounded, their jumps are also bounded. Therefore, there exists a constant M such that for all $k = 1, \ldots, n$, we have $U_{\tau_k}(\zeta_k) = U_{\tau_k}(\zeta_k) \wedge M$. This gives that

$$\mathbb{E} \int_0^T [U_t(e) - U_t(e) \wedge M]^2 \lambda(de) dt = 0 ,$$

and hence $U \le M$, $dt \otimes d\lambda \otimes d\mathbb{P}$-a.e. From the envelope theorem, we easily check that, since C is compact and θ bounded, $\tilde{f}$ satisfies Assumption 3.3.3. From Theorem 3.3.3, we get $(Y^1, Z^1, U^1) = (Y^2, Z^2, U^2)$.

It remains to show that $R^{(\pi)}$ is a supermartingale for any $\pi \in \mathcal{A}$. Since $\pi \in \mathcal{A}$, the process $\mathcal{E}(M^{(\pi)})$ is a positive local martingale, because it is the Doleans-Dade exponential of a local martingale whose the jumps are superior to -1. Hence, there exists a sequence of stopping times $(\vartheta_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \to \infty} \vartheta_n = T$, $\mathbb{P} - a.s.$, such that $\mathcal{E}(M^{(\pi)}_{t \wedge \vartheta_n})$ is a positive martingale for each $n \in \mathbb{N}$. The process $A^{(\pi)}$ is nondecreasing. Thus $R^{(\pi)}_{t \wedge \vartheta_n} = R_0 \mathcal{E}(M^{(\pi)}_{t \wedge \vartheta_n}) \exp(A^{(\pi)}_{t \wedge \vartheta_n})$ is a supermartingale, i.e. for $s \le t$

$$\mathbb{E}\big[R^{(\pi)}_{t \wedge \vartheta_n} \big| \mathcal{G}_s\big] \le R^{(\pi)}_{s \wedge \vartheta_n} .$$

For any set $A \in \mathcal{G}_s$, we have

$$\mathbb{E}\big[R^{(\pi)}_{t \wedge \vartheta_n} \mathbb{1}_A\big] \le \mathbb{E}\big[R^{(\pi)}_{s \wedge \vartheta_n} \mathbb{1}_A\big]. \tag{3.5.7}$$

On the other hand, since

$$R_t^{(\pi)} = -\exp\big(-\alpha\big(X_t^{x,\pi} - Y_t\big)\big),$$

we use both the uniform integrability of $(\exp(-\alpha X_\vartheta^{x,\pi}))$ where ϑ runs over the set of all stopping times and the boundedness of Y to obtain the uniform integrability of $(R^{(\pi)}_{\cdot\wedge\vartheta^n})$. Hence, the passage to the limit as n goes to ∞ in (3.5.7) is justified and it implies

$$\mathbb{E}\big[R_t^{(\pi)}\mathbb{1}_A\big] \leq \mathbb{E}\big[R_s^{(\pi)}\mathbb{1}_A\big].$$

This implies the claimed supermartingale property of $R^{(\pi)}$.

To complete the proof, we justify that the strategy $\hat{\pi}$ defined as the minimum argument of the driver of BSDE (3.5.4) is an optimal strategy. By definition of $\hat{\pi}$, we have $A^{(\hat{\pi})} = 0$ and hence, $R_t^{(\hat{\pi})} = R_0\mathcal{E}(M_t^{(\hat{\pi})})$ is a true martingale, since $\hat{\pi}$ is in $\mathcal{A}$, thanks to Lemma 3.5.1. As a result,

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\big(R_T^{(\pi)}\big) = R_0 = V^B(x).$$

Using that (Y, Z, U) is the unique solution of the BSDE given by (f, B), we obtain the expression (3.5.3) for the value function. $\square$

Part III

BID-ASK SPREAD MODELING, A PERTURBATION APPROACH

Chapter 4

Bid-Ask spread modeling, a perturbation approach

Joint paper with Vathana Ly Vath and Simone Scotti.

Abstract: Our objective is to study liquidity risk, in particular the so-called "Bid-Ask spread", as a by-product of market uncertainties. "Bid-Ask spread", and more generally "Limit order books" describe the existence of different sell and buy prices, which we explain by using different risk aversions of market participants. The risky asset follows a diffusion process governed by a Brownian motion which is uncertain. We use the error theory with Dirichlet forms to formalize the notion of uncertainty on the Brownian motion. This uncertainty generates noises on the trajectories of the underlying asset and we use these noises to expound the presence of Bid-Ask spreads. In addition, we prove that these noises also have direct impacts on the Mid-price of the risky asset. We further enrich our studies with the resolution of an optimal liquidation problem under these liquidity uncertainties and market impacts. To complete our analysis, some numerical results will be provided.

Keywords: Liquidity risk, Bid-Ask spread, error theory, portfolio selection, dynamic programming principle, tracker, Black-Scholes model, CEV model.

4.1 Introduction

Classical market models in mathematical finance assume infinite liquidity or perfect elasticity of traded assets. By liquidity, we mean market liquidity which corresponds to the ability for investors to act as a price taker, so that they buy and sell the assets with arbitrary volumes without changing the prices. They further assume that traders may buy or sell stocks at the same price, in other words, there is no difference between the Bid and Ask prices, which are both equal. However, as shown in the market microstructure literature, it is clear that large trades move the price of the assets. More specifically, the spread between the Bid and Ask prices does exist and is intrinsic to the financial market structure.

Relaxing this assumption is very important in the study of option hedging, optimal allocation and liquidation problems. It is particularly important for intraday trading and when dealing in many markets where the transactions frequency and/or the number of operators is low. The market liquidity crunch — Brunnermeie and Pedersen [30] and Brunnermeie [31] — we have witnessed during the recent financial crisis is a case in point. It was indeed the liquidity crunch which triggered a complete meltdown in the financial markets.

The main goal in the study of liquidity risk is to find the best way to quantify the costs incurred by investors and to understand how their trades may impact the prices dynamics of the traded assets. There are mainly three approaches in the modeling of liquidity cost and impact. The first approach is the use of impact functions as a way to replicate the dependencies of assets prices on the trading strategies. The trading impact on the price dynamics could be either permanent, for instance for large investors, see Frey [60], Platen and Schweizer [112], He and Mamaysky [65] and Ly Vath, Mnif and Pham [95], or temporary, e.g. for small investors who are mainly price-takers, see Cetin, Jarrow and Protter [34], Cetin and Rogers [35] and Cetin, Soner and Touzi [36]. The second approach is to fully consider the very structure of the market: the modeling of limit orders book, see for example Alfonsi, Schied and Schulz [1] and Cont, Stoikov and Talreja [37]. The third approach is the Bid-Ask spread modeling which partially takes into account the market structure. Proportional transaction cost could be considered as the most simple Bid-Ask spread model. This Bid-Ask spread component is generally combined with impact functions. For instance, Kharroubi and Pham [82] and Schied and Schoneborn [122] study optimal portfolio problems with the presence of both Bid-Ask spread component and temporary price impact.

In the above studied models, the general approach could be described as follows: assuming the existence of liquidity costs and impacts, the authors postulate a model replicating their effects. However, to our knowledge, few studies in the fields of mathematical finance have attempted to model the financial and economic rationales behind the existence of the Bid-Ask spread. This is precisely the objective in this paper: study and explain the existence of the Bid-Ask spread, and more generally limit order book, as a by-product of market uncertainties.

It is well-known that an asset price is theoretically the discounted of expected future cash flows, which are random processes and must be estimated. Thus, the value of an asset is an estimation obtained under uncertainty and may therefore be represented by a random variable. In other words, at any fixed time, the value of an asset is not observable but a random variable where its law or at least its mean value and variance may be characterized.

In conformity with this point of view, the presence of many sell and buy order prices can be explained by different risk aversions of market participants. In order to clarify this idea, we consider a "representative" price setter-market participant who has to place both a buy and a sell limit orders, i.e. the prices and the number of shares he is willing to buy and sell. Prior to setting the buy and sell orders, he obtains the distribution of possible asset values from the market information but has no possibility to observe the asset's realized values. A rational decision is to send a limit buy (sell) order with a price lower (higher) with respect to the asset mean value such that their difference justifies the risk taken. Of course, he adjusts those prices by increasing them if he runs short of stock, or cutting them if he starts accumulating excessive stocks.

The mathematical formulation of such problems relies on the specification of a coherent framework to describe the remaining randomness on prices. In our study, the asset value must depend on two random sources: the first one describes the evolution of the asset mean value while the second delineates the shape of asset (sell-buy) prices at a given fixed time. The coupling of the two probability spaces, with its respective filtration, requires complex tools and represents the principal drawback of this kind of approach. Therefore, we choose a different strategy based on error theory using Dirichlet forms formalism developed by Bouleau in [24], [25], [26] and [27]. The advantages of this approach are inherent to its elasticity and powerful tools. Order book framework justifies automatically many assumptions of error theory, e.g. Bid-Ask spreads are almost always very negligible with respect to the Mid-price, allowing the limit expansion approach.

Such an approach provides us with a perfect knowledge on the Bid-Ask spread component of the order book, i.e. the best/highest Bid price and the best/lowest Ask price of the order book. Once our Bid-Ask spread model obtained, as in Bertsimas and Lo [11], Almgren and Chriss [2], Obizhaeva and Wang [101] or [1], we investigate an optimal liquidation problem for a large portfolio. In order to completely solve this problem, in addition to the knowledge on the Bid-Ask spread component, one should equally consider the depth of the market. One way to consider the market depth is to model the limit order book with general shape functions as in [1], another is to consider an impact function as in [95]. In our study, we consider both aspects by combining the Bid-Ask spread component with an impact function to characterize the illiquidity of the market.

The article is organized as follows. In Section 2, we introduce the economic model for Bid-Ask spread and we present the analysis of prices variance and bias. In Section 3, we study an optimal liquidation problem associated with the Bid-Ask spread model developed in the previous section. And finally, in Section 4, we provide some numerical results.

4.2 The model

In this section, we aim at modeling the dynamics of the Ask and Bid prices. Our objective is to define an asset price model that considers the Bid-Ask spread as an inherent part of asset price evolution.

4.2.1 Theoretical analysis of path sensitivity and approximation

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process X representing an observable but non-traded benchmark index or asset which is governed by the following stochastic differential equation (SDE)

$$dX_t = r\, X_t\, dt + \sigma(t,\, X_t,\, \omega)\, X_t\, dW_t\,, \tag{4.2.1}$$

where r is the drift and σ is a function on $\mathbb{R}^+ \times \mathbb{R} \times \Omega$ that verifies the following assumptions:

Assumption 4.2.1. (Underlying diffusion)

1. SDE (4.2.1) admits a unique strong solution, denoted X_t, such that X_t is square-integrable and does not explode in finite time with probability 1.

2. The solution X_t of SDE (4.2.1) is always positive.

3. $f(x) = x\, \sigma(t,\, x,\, \omega)$ is a twice derivable function in x and the derivatives are Lipschitz and bounded.

4. The dependency of $\sigma(t,\, X_t,\, \cdot)$ with respect to the third variable is independent to the filtration $\mathcal{F}_t = \sigma(X_s,\, s \leq t)$, for all $t \in [0,\, T]$.

This Assumption 4.2.1 covers a large class of stochastic models in finance. In particular, Assumption 4.2.1 is satisfied by log-normal diffusion, a large part of local volatility models, see Dupire [51], and stochastic volatility models, for instance see Hull and White [68]. For Constant Elasticity of Variance model, we may refer to Cox [39].

To simplify our notation, we denote the first and second derivatives of $x\, \sigma(t,\, x,\, \omega)$ as

$$\begin{cases} \zeta(t,\, x,\, \omega) = \sigma(t,\, x,\, \omega) + x \dfrac{\partial \sigma}{\partial x}(t,\, x,\, \omega), \\ \eta(t,\, x,\, \omega) = 2 \dfrac{\partial \sigma}{\partial x}(t,\, x,\, \omega) + x \dfrac{\partial^2 \sigma}{\partial x^2}(t,\, x,\, \omega). \end{cases}$$

An increasing role of the assets management industry is the provide investors with investment tools capable of replicating a wide range of indices such as CAC 40, EURO STOX 50 or Real Estate Indices. These investment tools are often known as trackers. They are traded in the market like any other quoted assets but most of them are in general illiquid.

We assume that this illiquid asset price follows the same SDE as the tracked index, its Brownian motion is perturbed by the problem of replication. Therefore, the illiquid asset price is given by

$$dS_t = r\, S_t\, dt + \sigma(t,\, S_t,\, \omega)\, S_t\, dB_t\,, \tag{4.2.2}$$

where B_t is a Brownian motion, which is almost explained by W_t but characterized by a small uncertainty. In order to clarify our hypothesis, we assume that

$$B_t = \sqrt{e^{-\epsilon}}\, W_t + \sqrt{1 - e^{-\epsilon}}\, \hat{W}_t\,, \tag{4.2.3}$$

where ϵ is a small parameter and $(\hat{W}_t)$ is a Brownian motion, independent w.r.t $\mathcal{F}$, that resume all hedging errors.

The two Brownian motions, W_t and $\hat{W}_t$, play different roles. W_t describes the market information, that is progressively known through the index value. Therefore, at time t the information $\mathcal{F}_t = \sigma(X_s, s \leq t)$ is known, whereas the information $\mathcal{G}_t = \sigma(\hat{W}_s, s \leq t)$ is unknown or unobservable.

It is possible to compute directly the impact of the perturbation by using filtering theory, see for example Bain and Crisan [4] and Pham and Quenez [110]. However, due to the extreme complexity of the equations, we choose to follow a different approach using error theory. Indeed, in our analysis, we apply the error theory by using Dirichlet forms developed by Bouleau [24], [25], [26] and [27]. We fix an error structure $(\Omega,\, \mathcal{H},\, \mathbb{P},\, \mathbb{D},\, \Gamma)$, where $(\Omega,\, \mathcal{H},\, \mathbb{P})$ is the Wiener space in which the Brownian motion B lives, while Γ is an Ornstein-Uhlenbeck carré du champ operator with constant weight θ (see Section 3 in [26]). Using this theory, formula (4.2.3), known as Mehler formula, is automatically justified, see Section VI.2. in [25].

Error theory enables us to find a limited expansion of the law of the price of illiquid asset due to the noise on Brownian motion. In particular, we have the following results.

Theorem 4.2.1. (Law of illiquid asset price)

Under Assumption 4.2.1, the uncertainty on Brownian motion is transmitted to the stochastic process S, which represents the illiquid asset price. Then, any realization $\bar{\omega}$ of process X, at time t, fixes a random variable $S_t(\bar{\omega})$ described by

$$S_t(\bar{\omega},\, \hat{\omega}) = X_t(\bar{\omega}) + \epsilon\, \mathcal{A}[S_t](\bar{\omega}) + \sqrt{\epsilon\, \Gamma[S_t](\bar{\omega})}\, \tilde{\mathcal{N}}(\hat{\omega})\,, \tag{4.2.4}$$

where $\tilde{\mathcal{N}}$ is a centered reduced gaussian random variable independent w.r.t $\mathcal{F}_t$, while $\Gamma[S_t](\bar{\omega})$ and $\mathcal{A}[S_t](\bar{\omega})$ are given by

$$\begin{cases} \Gamma[S_t] = \theta\, M_t^2 \displaystyle\int_0^t \frac{X_s^2\, \sigma^2(s,\, X_s,\, \omega)}{M_s^2}\, ds + \Gamma[S_0]\, M_t^2\,, \\ \mathcal{A}[S_t] = M_t \displaystyle\int_0^t \frac{\eta(s,\, X_s,\, \omega)\, \Gamma[S_s] - \theta\, X_s\, \sigma(s,\, X_s,\, \omega)}{2\, M_s} \big[dW_s - \zeta(s,\, X_s,\, \omega)\, ds\big]\,, \\ M_t = \mathcal{E}\Big\{ \displaystyle\int_0^t \zeta(s,\, X_s,\, \omega)\, dW_s + r\, t\Big\}\,, \end{cases} \tag{4.2.5}$$

where $\mathcal{E}$ denotes the Doleans-Dade exponential.

Proof. The proof of this theorem is mainly based on the truncated expansion in error theory using Dirichlet forms, see [25] and [26]. The two following Lemmas 4.2.1 and 4.2.2 form the main backbone of the proof. Indeed, they give the expression of the variance $\Gamma[S_t]$ and the bias $\mathcal{A}[S_t]$. □

Lemma 4.2.1. (Variance due to Brownian motion)

Let X be the solution of SDE (4.2.1) and assume that Assumption 4.2.1 holds. Then, the uncertainty effect on process S satisfies the following SDE

$$d\Gamma[S_t] = 2\, \zeta(t,\, X_t,\, \omega)\, \Gamma[S_t]\, dW_t + \big[2\, r + \zeta^2(t,\, X_t,\, \omega)\big]\Gamma[S_t]\, dt + \theta\, \sigma^2(t,\, X_t,\, \omega)\, X_t^2\, dt\,. \tag{4.2.6}$$

Moreover, $\Gamma[S_t]$ has the following closed form

$$\Gamma[S_t] \;=\; \theta\, M_t^2 \int_0^t \frac{X_s^2\, \sigma^2(s,\, X_s,\, \omega)}{M_s^2}\, ds + \Gamma[S_0]\, M_t^2.$$

Proof. The proof of this lemma is postponed in Appendix 4.5.1.

□

Lemma 4.2.2. (Bias due to Brownian motion)

Let X be the solution of SDE (4.2.1) and assume that Assumption 4.2.1 holds. Then, the bias effect on process S satisfies the following SDE

$$d\mathcal{A}[S_t] = r\mathcal{A}[S_t]\, dt + \Big[\zeta(t,\, X_t,\, \omega)\mathcal{A}[S_t] + \frac{1}{2}\eta(t,\, X_t,\, \omega)\, \Gamma[S_t] - \frac{\theta}{2}\sigma(t,\, X_t,\, \omega)\, X_t\Big]\, dW_t\,. \tag{4.2.7}$$

Moreover, $\mathcal{A}[S_t]$ has the following closed form

$$\mathcal{A}[S_t] \;=\; M_t \int_0^t \frac{\eta(s,\, X_s,\, \omega)\, \Gamma[S_s] - \theta\, X_s\, \sigma(s,\, X_s,\, \omega)}{2\, M_s} \big[dW_s - \zeta(s,\, X_s,\, \omega)\, ds\big].$$

Proof. The proof of this lemma is equally postponed in Appendix 4.5.1.

□

Remark 4.2.1. (Closed forms)

Equations (4.2.5) show an interesting property of processes $\Gamma[S_.]$ and $\mathcal{A}[S_.]$, it is easy to check that the law of $(\Gamma[S_t], \mathcal{A}[S_t])$ is completely explicit given the law of the triplet $(X_t, W_t, \sigma(t, X_t, \cdot))$. Therefore, Equations (4.2.5) are closed forms in the sense of involving only algebraic operations and stochastic integrals.

Remark 4.2.2. (Black-Scholes case)

In the particular case of σ constant, i.e. in the Black-Scholes model, Equations (4.2.5) are simplified with $\Gamma[S_t]$ proportional to the square of X_t and $\mathcal{A}[S_t]$ proportional to X_t.

Moreover, we have the following corollary:

Corollary 4.2.1. (Equilibrium price)

The equilibrium price, i.e. the mean of the price distribution, is given by

$$S_t^M(\bar{\omega}) = \mathbb{E}\big[S_t(\bar{\omega}, \hat{\omega}) \,\big|\, \mathcal{F}_t\big] = X_t(\bar{\omega}) + \epsilon\, A[S_t](\bar{\omega}). \tag{4.2.8}$$

The equilibrium price is therefore different from X_t. In particular, this shift exists in Black-Scholes framework. However, in this case, this shift is proportional to X_t, so it is possible to include it into the starting point S_0^M. This shift can explain tracking errors usually remarked on ETF-markets, for instance see Frino and Gallagher [62].

Finally as a corollary of the two previous lemmas, we have the following Markov property:

Corollary 4.2.2. (Markov property)

The triplet $\tilde{X}_t = (X_t, \Gamma[S_t], \mathcal{A}[S_t])$ is a markovian process if and only if X_t is markovian.

This assertion is a direct consequence of the fact that $\Gamma[S_t]$ verifies SDE (4.2.6) which only depends on process X, and finally $\mathcal{A}[S_t]$ follows SDE (4.2.7) which depends on both X_t and $\Gamma[S_t]$.

4.2.2 Bid-Ask model

Theorem 4.2.1 gives us the law of the illiquid asset price given the value of the benchmark/index. In this subsection, we explain how this approach can be used to define Bid and Ask prices and suggest a model that reproduces it.

We consider the presence of many agents on the market, all informed about the economic evolution of the benchmark price X, but without money-market intelligence about the residual information drawn by the perturbation, i.e. the independent Brownian motion $\hat{W}$. We assume that all agents are risk adverse and can estimate the distribution of the uncertainty of the illiquid asset price, at any fixed time t, given by Theorem 4.2.1. We now consider uniquely price-setter agents or liquidity providers who place limit orders as opposed to market orders placed by price-taker agents, liquidity takers. Indeed, their aggregated limit orders constitute an order book and therefore the Bid-Ask spread. It stands to reason that, at any given time t, there exists a price-setter agent with minimal risk aversion with respect to other agents. This agent accepts to buy the asset at a price S_t^B bigger than the prices proposed by the other agents. Thus, the price proposed by this agent is the Bid price and it is denoted by S_t^B. This price is completely defined by the law of the illiquid asset and the risk aversion of this agent. A symmetric analysis generates the Ask price S_t^A.

Let us assume, for sake of simplicity, that there exists a representative price-setter agent who always submits the best buy and sell prices, which we respectively define as best Bid price S_t^B and best Ask price S_t^A. Indeed, we assume that he accepts to buy the illiquid asset at a price S_t^B such that the risk of overvaluing of this asset is equal to a supportable risk probability χ_B. Therefore he takes the risk against the expected earnings, see Figure 4.1. In conclusion, S_t^B is the χ_B-quantile of the illiquid asset price distribution given by the uncertainty on the Brownian motion, see Theorem 4.2.1.

The definition of Ask price is symmetric, i.e. S_t^A is the $(1-\chi_A)$-quantile of the illiquid asset price distribution given by the uncertainty on the Brownian motion. It is clear that $\chi_A + \chi_B < 1$.

Definition 4.2.1. (Static Bid and Ask prices)

Let χ_B and χ_A with $\chi_A + \chi_B < 1$ be risks taken by the "representative price setter" in respectively overvaluing and undervaluing the illiquid asset at a given time t. The corresponding Bid S_t^B and Ask S_t^A prices are defined as follows

$$\begin{cases} S_t^B = X_t + \epsilon\,\mathcal{A}[S_t] + \sqrt{\epsilon\,\Gamma[S_t]}\;\tilde{\mathcal{N}}^{-1}(\chi_B)\,, \\ S_t^A = X_t + \epsilon\,\mathcal{A}[S_t] + \sqrt{\epsilon\,\Gamma[S_t]}\;\tilde{\mathcal{N}}^{-1}(1-\chi_A)\,. \end{cases}$$

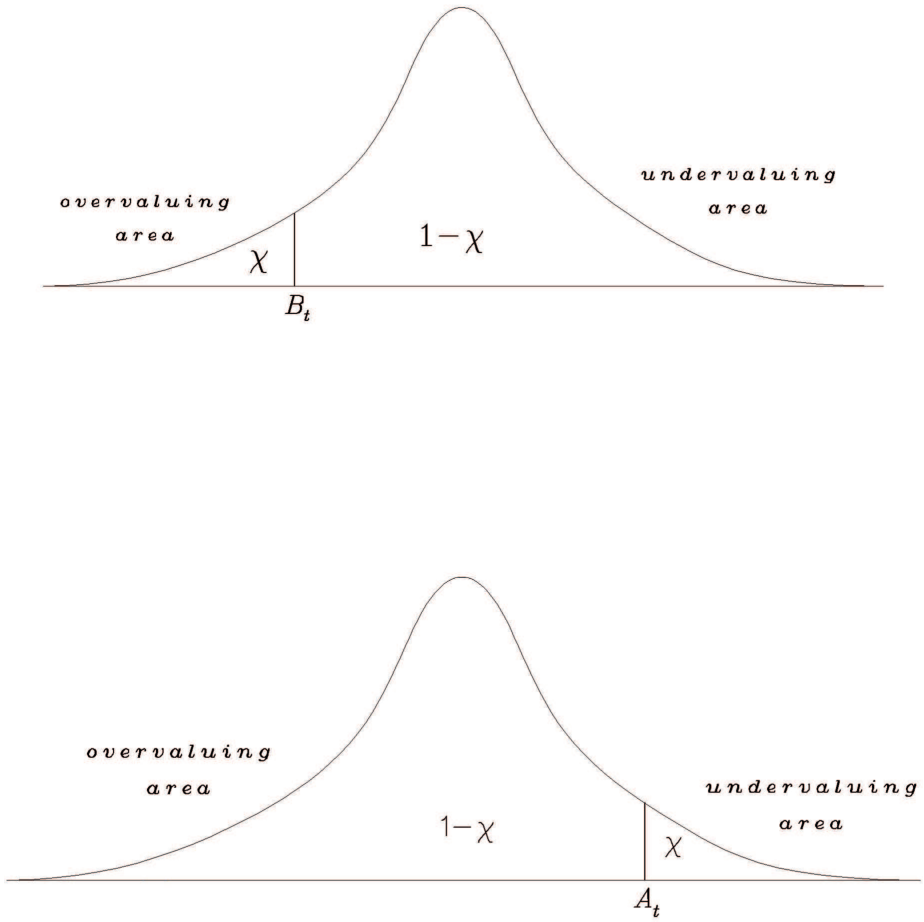

Figure 4.1: Bid and Ask prices definition, defined by a risk probability χ

Since the law of residual uncertainty is always gaussian, the definition of the supportable risk is equivalent to the definition of the trader utility function. For sake of simplicity, we fix the same supportable risk for sell S_t^B and buy S_t^A prices, i.e. $\chi_B = \chi_A = \chi$. In Figure 4.2, we present an example. We consider the Constant Elasticity of Variance model, we choose a trajectory of X and we can compute the evolution of the Bid price S^B, the Mid-price S^M and the Ask price S^A.

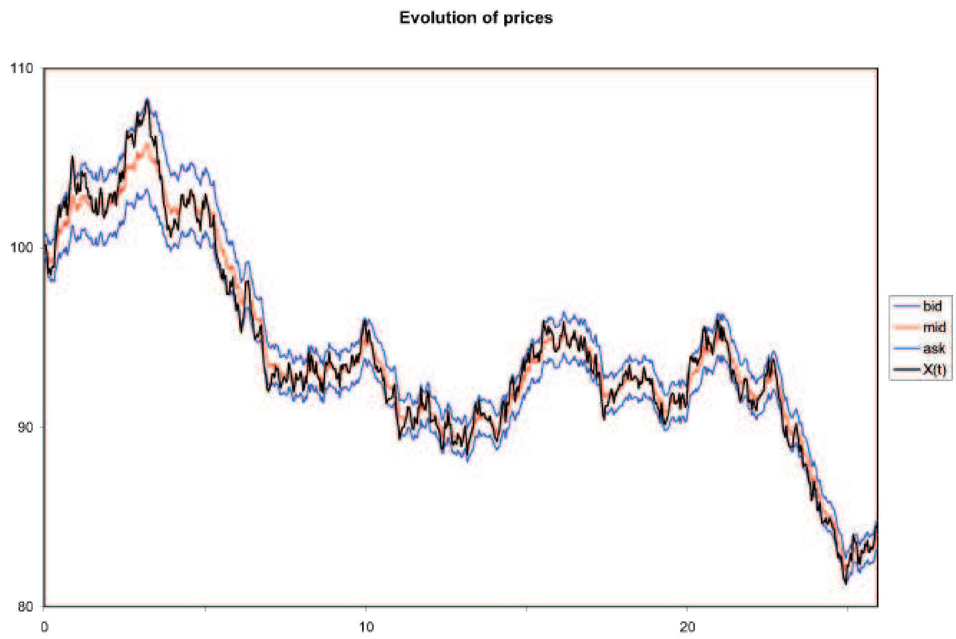

Figure 4.2: An example with Constant Elasticity of Variance model: we choose a single path of the process X (in black) and we can compute explicitly the Mid-price (in red) and the Bid and the Ask prices using a standard deviation.

Remark 4.2.3.

The trajectories of X and of S^M are different. This is due to the fact that X is not linear w.r.t W in CEV diffusion, see Section 4.3.3, hence the error introduces a bias, see Corollary 4.2.1.

In order to define a Bid-Ask model, we have to choose a dynamics for this risk aversion, since a static risk aversion is very restrictive. The dynamic risk aversion is not only justified by very nature of the "representative price-setter agent" but also by market orders flow from price-taker agents.

We now turn to the choice of a dynamics of the Bid-Ask spread. In the economic literature, Bid-Ask spread depends mainly on two factors: the value of the stock and the trading volume, see Potters and Bouchaud [114], Bialkowski *et al.* [13] and Lehalle [89].

In particular, the Bid-Ask spread converges to zero (resp. infinity) when the asset price goes to zero (resp. infinity). However the relative spread[1] converges to a strictly positive constant when the asset price goes to zero and converges to zero when the asset price goes to infinity. This effect can be explained endogenously with the evolution of the variance $\Gamma[S_t]$, see Section 4.3.3 for an analysis in CEV case.

The trading volume equally plays a leading role. If we analyze two assets with almost the same price but with different trading volumes, we notice that the lower the trading volume, i.e the more illiquid is the asset, the larger Bid-Ask spread, see for instance Wang and Yau [127]. An economic explanation is that the traders accept higher risks if they can easily close their positions, which is possible with the presence of many counterparts. Historical data show that average trading volumes are mean-reverting on medium term, Bid-Ask spread shows the same behavior.

In order to fit this behavior, we use an Ornstein-Uhlenbeck process Y or more precisely the exponential of an Ornstein-Uhlenbeck process $Z_t = \exp(Y_t)$.

We consider an Ornstein-Uhlenbeck process Y with the following SDE

$$dY_t = -\phi\, Y_t\, dt + \sigma_Y\, dW_t^Y\,, \tag{4.2.9}$$

where ϕ and σ are positive parameters and W_t^Y is a Brownian motion independent of $\mathcal{F}_t$. The process Y has the following closed form expression

$$Y_t = Y_0\, e^{-\phi\, t} + \sigma_Y \int_0^t e^{-\phi\,(t-u)} dW_u^Y\,. \tag{4.2.10}$$

We consider the following model of the Bid-Ask spread:

Definition 4.2.2. (Bid and Ask model)

At any time t, given the value of the benchmark $X_t(\bar{\omega})$, the Bid and Ask prices are given by

$$\begin{cases} S_t^A = X_t + \epsilon\, \mathcal{A}[S_t] + \sqrt{\epsilon\, \Gamma[S_t]}\; Z_t\,, \\ S_t^B = X_t + \epsilon\, \mathcal{A}[S_t] - \sqrt{\epsilon\, \Gamma[S_t]}\; Z_t. \end{cases} \tag{4.2.11}$$

Remark 4.2.4.

The choice of this model is justified by the following properties:

- *Positivity*: the Ask price is always bigger than the Bid price.

- *Closed forms*: in our model, all terms, excepted the underlying X_t, have an explicit form. The law of X_t is the unique law that we have to estimate numerically. This computation can be easily performed using a Monte-Carlo method.

[1]The relative spread is defined as the ratio between Bid-Ask spread and the asset.

- *Error tracking*: the Mid-price S_t^M is different to the benchmark one, since a systematic bias exists. The two prices are relatively closed given a small parameter ϵ.
- *Separation*: in our model, the Bid-Ask spread is explained by two independent factors. The first factor concerns the sensitivity of the benchmark/index level path with respect to the Brownian motion W, which, in an economic point of view, corresponds to the sensitivity with respect to "market" information. The second one is risk aversion of market participants mainly depending on trading volumes.
- *Mean reverting*: if the value of underlying is relatively stable, the Bid-Ask spread shows a mean reverting behavior.
- *Bid-Ask spread tails*: given the evolution of the benchmark, the law of the Bid-Ask spread is lognormal, so extremely wide or small spreads are possible but with a very low probability.

4.3 Optimal liquidation portfolio problem

4.3.1 The economic motivations and the objective functions

Given the above Bid-Ask spread model, as defined in Definition 4.2.2, which highlights the market imperfections due to liquidity risk, a natural but challenging problem to both professional and academic in finance to solve is the optimal portfolio liquidation problem. Let us consider a price-taker investor who decides to close his position over a finite horizon, he has to define a trading strategy which maximizes his terminal portfolio value. Since the attempt to sell the whole block of shares causes, generally, a lack of balance between supply and demand, thus, resulting in an average selling price well below the best pre-order Bid price. In practice, large orders are generally slit into a number of consecutive small orders to reduce the overall price impact.

Let us therefore investigate a problem of an investor seeking to liquidate N shares of stock over a finite time horizon T. To solve this problem, we consider a discrete framework by assuming that trading occurs only at discrete times $t_1 < t_2 < ... < t_n = T$. A strategy decision π for the investor is a sequence $(\pi_i)_{1\leq i\leq n}$ valued in $[0, N]$ where π_i is $\mathcal{F}_{t_i}$-measurable and represents the number of shares to be sold at time t_i. We define an admissible strategy as being a strategy π such that $\sum_{i=1}^n \pi_i = N$. As such we define the set of admissible strategies $\mathcal{A}(t_i, p)$ as

$$\mathcal{A}(t_i,p) = \Big\{\pi = \{\pi_i,\dots,\pi_n\},\ \pi_j \geq 0 \ \ \forall\, j \in \{i,\dots,n\} \text{ and } \sum_{j=i}^{n} \pi_j = p\Big\}. \tag{4.3.12}$$

Price impact. In addition to the existence of the Bid-Ask spread as evidenced in the previous section, we equally take into account a lack of market depth by assuming that marginal selling prices are non-increasing. Indeed, there is no infinite liquidity at either the best Bid price nor at the best Ask price. For that purpose, we introduce an impact function g which indicates the average price obtained at a sell market order. More precisely, when an investor submits a sell order of x number of shares through a single sell order at time t, the obtained average price $\bar{S}_t^B(x)$ is assumed

$$\bar{S}_t^B(x) = S_t^B \, g(x) \, , \tag{4.3.13}$$

where the function $g(\cdot)$ verifies the following assumption:

Assumption 4.3.2. (Trading impact function)

1. $g(\cdot)$ is a continuous positive deterministic function independent to S_t.

2. $g(\cdot)$ is non-increasing.

3. $h(x) = x \, g(x)$ is strictly non-decreasing and concave.

Remark 4.3.5.

1. We assume that trading impact is temporary when trading occurs. Only the price takers who place market price orders (at best selling prices) pay the liquidity costs. After the trades, the order book is filled back with limit orders from other market participants [1].

2. $g(x)$ corresponds to the ratio between the average stock price received following the sale of x shares at market price order and the best Bid price. This average price obviously decreases with the number of traded shares.

3. The marginal price $[(x + \delta x)g(x + \delta x) - xg(x)]S_{t_i}^B$ should be non-negative and non-increasing. Therefore, function $h(x) = x \, g(x)$ must be non-decreasing and concave. The concavity comes from the shape of the order book, which displays a maximum around the best Bid price, see Potters and Bouchaud [114].

Objective function. The objective of the investor is to maximize its terminal wealth from the sales of the stock shares in holding. To fully describe our state process, we should take into account not only the processes X and Y but also $\Gamma[S_.]$ and $A[S_.]$. As such, the state process to consider is $Z = (\tilde{X}, Y)$, where $\tilde{X}$ is defined as in Corollary 4.2.2. At any initial time t_i and any state value $(z, \, p)$ of the variables $(Z_{t_i}, \, P_{t_i})$, with P_{t_i} the number of stock shares that we initially have at time t_i, we define our reward function for any strategy $\pi \in \mathcal{A}(t_i, p)$ by

$$J(i, \, z, \, p, \, \pi) = \mathbb{E}\Big[\sum_{j=i}^{n} e^{-\rho(t_j - t_i)} \pi_j \, S_{t_j}^B \, g(\pi_j) \Big] ,$$

with ρ representing the interest rate, $\mathcal{A}(t_i, p)$ the set of admissible strategies defined as (4.3.12) and $S^B_{t_j}$ the Bid price as defined in Definition 4.2.11.

The objective of the investor is to maximize this reward function over all admissible strategies. We therefore introduce the following value function

$$v(i,\, z,\, p) = \sup_{\pi \in \mathcal{A}(t_i, p)} (J(i,\, z,\, p, \pi)). \tag{4.3.14}$$

For an initial state $(i,\, z,\, p)$, $\hat{\pi} \in \mathcal{A}(t_i, p)$ is called an optimal strategy if

$$v(i,\, z,\, p) = J(i,\, z,\, p,\, \hat{\pi}).$$

In the sequel, we restrict the set of admissible strategies $\mathcal{A}(t_i, p)$ to Markov strategies subset of $\mathcal{A}(t_i, p)$, which is denoted $\bar{\mathcal{A}}(t_i, p)$ (that is possible from Proposition 8.1 of [10]).

4.3.2 Theoretical solution of the optimization problem

We now prove the existence of a solution to our optimization problem (4.3.14) and its uniqueness. Given an initial N stock shares of the risky asset, our objective is to prove that an optimal strategy in liquidating our portfolio exists in $\mathcal{A}(t_1, N)$ and it is unique. For notation convenience, we shall denote Z_i (resp. S_i, P_i) for Z_{t_i} (resp. S_{t_i}, P_{t_i}). Using the dynamic programming principle, we have:

Theorem 4.3.2. (Existence)

Under Assumptions 4.2.1 and 4.3.2, there exists an optimal policy $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)$ to the optimization problem, such that $\hat{\pi} \in \bar{\mathcal{A}}(t_1, N)$. This optimal strategy is given by the argmax in the following programming equation

$$\begin{cases} v(i,\, z,\, p) = \operatorname*{ess\,sup}_{0 \le \pi_i \le p} \Big\{ \pi_i\, s^B_i\, g(\pi_i) + \mathbb{E}\Big[e^{-\rho(t_{i+1}-t_i)} v(i+1,\, Z^{i,z}_{i+1},\, p-\pi_i) \Big| \mathcal{F}_{t_i} \Big] \Big\}, \\ v(n,\, z,\, p) = p\, s^B_n\, g(p), \end{cases} \tag{4.3.15}$$

where s^B_i is defined by the components of the variable z_i (see the definition of S^B in (4.2.11)).

Proof. This is an immediate application of Proposition 8.5 of [10]. From (4.2.11), we have

$$\mathbb{E}\big[S^B_t\big] = \mathbb{E}[X_t] + \epsilon\, \mathbb{E}[\mathcal{A}[S_t]] - \mathbb{E}\big[\sqrt{\epsilon\, \Gamma[S_t]}\big]\, \mathbb{E}\big[e^{Y_t}\big].$$

From Assumption 4.2.1, X does not explode in finite time and $\sigma(t,\, X_t,\, \omega)$, $\zeta(t,\, X_t,\, \omega)$ and $\eta(t,\, X_t,\, \omega)$ are Lipschitz and bounded, thus $\mathbb{E}[\mathcal{A}[S_t]] < \infty$ and $\mathbb{E}\big[\sqrt{\epsilon\, \Gamma[S_t]}\big] < \infty$. Since the process Y is an Ornstein-Uhlenbeck, it is clear that $\mathbb{E}[e^{Y_t}] < \infty$ and as the process X

is square integrable, we also have that $\mathbb{E}[X_t] < \infty$. Therefore $\mathbb{E}[S_t^B] < \infty$ and it enables us to check Assumptions (F^+) and (F^-) in Proposition 8.5 of [10]. Then, it remains to prove that the supremum in relation (4.3.15) is attained. This immediately follows from the continuity of $v(i+1,\, z,\, p)$ with respect to p, which is the case thanks to Assumption 4.3.2. □

We now turn to the uniqueness property of the optimal strategy:

Theorem 4.3.3. (Uniqueness)

Under Assumptions 4.2.1 and 4.3.2, there is at most one solution to optimization problem (4.3.14).

Proof. We first introduce the following function ϑ defined for any $x \leq y$ as

$$\vartheta(i,\, z,\, x,\, y) \;=\; x\, S_i^B\, g(x) + \mathbb{E}\Big[e^{-\rho\Delta i} v\big(i+1,\, Z_{i+1}^{i,z},\, y-x\big)\Big|\mathcal{F}_{t_i}\Big],\; i \in \{1,\ldots,n-1\},$$

with $\Delta i := t_{i+1} - t_i$.

We now prove by iteration that ϑ is concave with respect to the third and fourth variables (x, y) and the value function v is strictly concave with respect to the last variable p, i.e. $v(i,\, z,\, p)$, defined in (4.3.15), is strictly concave with respect to p, for all $i \in \{1, ..., n\}$.

We first note that for $i = n$, $v(n,\, z,\, .)$ is strictly concave in the last variable, thanks to Assumption 4.3.2. We can easily verify that $\vartheta(n, ., ., .)$ is concave with respect to the third and fourth variables.

Assuming that for $i+1$, $v(i+1,\, z,\, p)$ and $\vartheta(i+1,\, z,\, x,\, y)$ are respectively strictly concave with respect to p and to $(x,\, y)$, let prove that it is equally the case for i.

Let $0 \leq \lambda \leq 1$, $(x_1,\, y_1)$ and $(x_2,\, y_2)$, with $0 \leq x_i \leq y_i \leq N$, we have

$$\begin{aligned}\vartheta(i,\, z,\, \lambda\, x_1 + (1-\lambda)\, x_2,\, \lambda\, y_1 + (1-\lambda)\, y_2) &= (\lambda\, x_1 + (1-\lambda)\, x_2)\, S_i^B\, g(\lambda\, x_1 + (1-\lambda)\, x_2)) \\ &\quad + \mathbb{E}\Big[e^{-\rho\Delta i} v(i+1,\, Z_{i+1}^{i,z},\, \lambda\,(y_1 - x_1) + (1-\lambda)\,(y_2 - x_2))\Big|\mathcal{F}_{t_i}\Big],\end{aligned}$$

since the first term is strictly concave, we have

$$(\lambda\, x_1 + (1-\lambda)\, x_2)\, S_i^B\, g(\lambda\, x_1 + (1-\lambda)\, x_2) > \lambda\, x_1\, S_i^B\, g(x_1) + (1-\lambda)\, x_2\, S_i^B\, g(\, x_2),$$

and by iteration the second term is strictly concave, we have

$$\begin{aligned}v(i+1,\, Z_{i+1}^{i,z},\, \lambda\,(y_1 - x_1) + (1-\lambda)\,(y_2 - x_2)) &> \lambda\, v(i+1,\, Z_{i+1}^{i,z},\, y_1 - x_1) \\ &\quad +(1-\lambda)\, v(i+1,\, Z_{i+1}^{i,z},\, y_2 - x_2).\end{aligned}$$

Taking the expectation, we get

$$\lambda\,\mathbb{E}\Big[e^{-\rho\Delta_i}v\big(i+1,\,Z^{i,z}_{i+1},\,y_1-x_1\big)\Big|\mathcal{F}_{t_i}\Big]+(1-\lambda)\,\mathbb{E}\Big[e^{-\rho\Delta_i}v\big(i+1,\,Z^{i,z}_{i+1},\,y_2-x_2\big)\Big|\mathcal{F}_{t_i}\Big]$$
$$\leq\ \mathbb{E}\Big[e^{-\rho\Delta_i}v\big(i+1,\,Z^{i,z}_{i+1},\,\lambda\,(y_1-x_1)+(1-\lambda)\,(y_2-x_2)\big)\Big|\mathcal{F}_{t_i}\Big]\,.$$

Thus $\vartheta(i,\,z,\,x,\,y)$ is strictly concave with respect to $(x,\,y)$, as the sum of two strictly concave functions.

We now prove that $v(i,\,z,\,y)$ is strictly concave with respect to y. For any $0\leq x_1\leq y_1$ and $0\leq x_2\leq y_2$, from the expression of v in (4.3.15), we have for all $0\leq\lambda\leq 1$

$$v(i,\,z,\,\lambda\,y_1+(1-\lambda)\,y_2)\geq\vartheta(i,\,z,\,\lambda\,x_1+(1-\lambda)\,x_2,\,\lambda\,y_1+(1-\lambda)\,y_2).$$

Since $\vartheta(i,\,z,\,x,\,y)$ is strictly concave with respect to $(x,\,y)$, we get

$$v(i,\,z,\,\lambda\,y_1+(1-\lambda)\ y_2)>\lambda\,\vartheta(i,\,z,\,x_1,\,y_1)+(1-\lambda)\,\vartheta(i,\,z,\,x_2,\,y_2).$$

The latter equality holds for any positive $x_1\leq y_1$ and $x_2\leq y_2$. In particular since the supremum is attained (from Theorem 4.3.2), we can take x_1^* and x_2^* such that

$$\begin{aligned}
v(i,\,z,\,y_1) &= \vartheta(i,\,z,\,x_1^*,\,y_1)=\sup_{0\leq x\leq y_1}\ \vartheta(i,\,z,\,x,\,y_1)\,,\\
v(i,\,z,\,y_2) &= \vartheta(i,\,z,\,x_2^*,\,y_2)=\sup_{0\leq x\leq y_2}\ \vartheta(i,\,z,\,x,\,y_2)\,,
\end{aligned}$$

thus

$$v(i,\,z,\,\lambda\,y_1+(1-\lambda)\,y_2)>\lambda\,v(i,\,z,\,y_1)+(1-\lambda)\,v(i,\,z,\,y_2).$$

Hence, $v(i,\,z,\,p)$ is strictly concave with respect to p. We have therefore proved the strict concavity of both functions. Using relation (4.3.15) and the above concavity property, we may obtain by iteration at most one solution to the optimization problem. □

4.3.3 Log-Normal and Constant Elasticity of Variance Diffusions

We now restrict our study to two particular diffusion models, with the first being the log-normal diffusion, i.e.

$$dX_t=r\,X_t\,dt+\sigma\,X_t\,dW_t\,.$$

It is plain that this diffusion verifies Assumption 4.2.1. In this case, we remark that the bias $\mathcal{A}[S_t]$ and the variance $\Gamma[S_t]$ become proportional respectively to X_t and X_t^2. As a result,

$$S_t^B=X_t\big[1-\epsilon\,a+\sqrt{\epsilon\,\gamma}\,e^{Y_{t_i}}\big]\,,\tag{4.3.16}$$

and the average price at which we sell a quantity π_i at time t_i, given by formula (4.3.13), is simplified and we have the following average price

$$\bar{S}^B_{t_i}(\pi_i) = X_{t_i}\, g(\pi_i)\, \left[1 - \epsilon\, a + \sqrt{\epsilon\, \gamma}\, e^{Y_{t_i}}\right], \tag{4.3.17}$$

where a and γ are constants.

As such, we consider the first historical extension of the Black-Scholes model, which is the constant elasticity of variance (CEV) model, see Cox [39] and Cox and Ross [40]. This extended model importantly takes into account the heteroscedasticity of the assets returns and explains the down-slopping behavior of the implied volatility, see for instance Macbeth and Merville [96].

Assumption 4.3.3. (CEV diffusion)

The volatility function $\sigma(t,\, X_t,\, \omega)$ is equal to $\sigma\, X_t^\alpha$, where σ is a positive constant and α is constant and belongs to $(-1,\, 1)$. We also assume that $X_t \geq \xi > 0$ for all $t \in [0,\, T]$. That is SDE (4.2.1) is replaced by the following SDE

$$dX_t = r\, X_t\, dt + \sigma\, X_t^{\alpha+1}\, dW_t\,. \tag{4.3.18}$$

The CEV diffusion, unfortunately, does not verify Assumption 4.2.1. However, all previous results still hold and their proofs remain substantially the same with the main difference coming from some properties of CEV diffusion that can be found in Jeanblanc *et al.* [73].

Under Assumption 4.3.3, we have also the following rewriting of Theorem 4.2.1.

Corollary 4.3.3. (Constant Elasticity of Variance model)

Under Assumption 4.3.3, the result of Theorem 4.2.1 remains true and equations (4.2.5) *are replaced by*

$$\begin{cases} \Gamma[S_t] = \theta\, M_t^2 \displaystyle\int_0^t \frac{\sigma^2\, X_s^{2\alpha+2}}{M_s^2}\, ds + \Gamma[S_0]\, M_t^2\,, \\ \mathcal{A}[S_t] = M_t \displaystyle\int_0^t \frac{\alpha\,(\alpha+1)\, X_s^{\alpha-1}\, \Gamma[S_s] - \theta\, \sigma\, X_s^{\alpha+1}}{2\, M_s}\, \left[dW_s - \sigma\,(\alpha+1)\, X_s^\alpha\, ds\right], \\ M_t = \mathcal{E}\Big\{\sigma\,(\alpha+1) \displaystyle\int_0^t X_s^\alpha\, dW_s + r\, t\Big\}\,. \end{cases} \tag{4.3.19}$$

Moreover, the martingale parts of the Doob decomposition of $\sqrt{\Gamma[S_t]}$ *and* $\frac{\sqrt{\Gamma[S_t]}}{X_t}$ *are respectively*

$$\begin{cases} \sigma\,(\alpha+1)\, X_t^\alpha\, \sqrt{\Gamma[S_t]}\, dW_t\,, \\ \sigma\, \alpha\, X_t^{\alpha-1}\, \sqrt{\Gamma[S_t]}\, dW_t\,. \end{cases} \tag{4.3.20}$$

Proof. The proof of the first part is just a simplification of relation (4.2.5) in the case of CEV diffusion. The second part is an easy application of Itô formula. □

Remark 4.3.6.
In particular, we may notice that in the CEV case:

- The fluctuations of the absolute spread, which are proportional to $\sqrt{\Gamma[S_t]}$, are always positive correlated with the underlying X_t.

- The fluctuations of the relative spread, which are proportional to the ratio $\sqrt{\Gamma[S_t]}$ over X_t, are negative (resp. positive) correlated with the underlying X_t if the CEV-exponent α is negative (resp. positive). The case usually treated in literature is when the CEV-exponent is smaller than one, for instance see Black [19]. Therefore, the relative Bid-Ask spread grows when the asset price falls, whereas the absolute Bid-Ask spread falls with the benchmark.

This remark is important, since it is well-known on financial markets that the Bid-Ask spread converges to zero (resp. infinity) when the asset price goes to zero (resp. infinity). Instead, the relative spread grows when the asset price goes to zero and converges to zero when the asset price goes to infinity. The previous remark said that our model can explained endogenously this effect with the evolution of the variance $\Gamma[S_t]$ if we suppose that the parameter α is negative, i.e. the diffusion is sub-linear. This case is usually presented in literature as a way to explain why the BS model overprices in-the-money calls options and underprices out-of-the-money ones, see for instance Cox and Ross [40] and Macbeth and Merville [96].

4.4 Numerical results

In this section, we provide some numerical results of the optimal strategy of the liquidation problem. For this purpose, we use two models

1. The Black-Scholes model with the drift equals zero. This model is used as a basic reference, since we have closed form expressions.

2. A CEV model with the drift equals zero and the CEV-exponential equals -0.7, and the initial value of underlying is assumed equal to 1. This model is used to evaluate the impact of sensitivity with respect to the Brownian motion.

For both models, we consider that the process Y follows an Ornstein-Uhlenbeck model with volatility equals 0.5 and mean-reverting parameter equals 0.02. We consider the following trading impact function g

$$g(x) = \exp(-\lambda x),$$

where the constant $\lambda < \frac{1}{N}$, with N the total number of shares to liquidate. It is rather clear that function g verifies Assumption 4.3.2 on the interval $[0,\, N]$. We restrict our function to the set $[0, N]$ given the fact that the optimal liquidation strategy abstains to buy shares at any time.

The optimal liquidation strategy is determined by using the dynamic programming equation (4.3.15). The classical approach to solve this kind of problem is to discretize all processes and to reduce the computation on a finite probability space (see for example Bardou *et al.* [6] or Barrera-Esteve *et al.* [8]). Thus, we discretize our processes using Monte-Carlo simulations for CEV and closed forms for log-normal and Ornstein-Uhlenbeck diffusions.

4.4.1 Black-Scholes case

We first consider the case when the underlying X follows SDE (4.2.1) with $\alpha = 0$, i.e. the Black Scholes model. The dynamic programming principle (4.3.15) gives us the optimal strategy to liquidate our portfolio. The strategy depends on three factors

- the level of the underlying X_t,
- the value of the liquidity process, i.e. the Bid-Ask spread,
- the residual quantity of stocks that we have to sell.

In Figure 4.3, 4.4, and 4.5, we present our numerical results in the Black-Scholes case, in particular, the dependencies of the optimal selling strategy on the three factors mentioned above.

Remark 4.4.7. (Black-Scholes case)

– We find that the optimal strategy is completely independent with respect to underlying X_t (see Figure 4.3). This result is coherent with the literature, see Alfonsi *et al.* [1], Almgren and Chriss [2] and Obizhaeva and Wang [101], given the fact that the spread is proportional to the underlying price in this particular case.

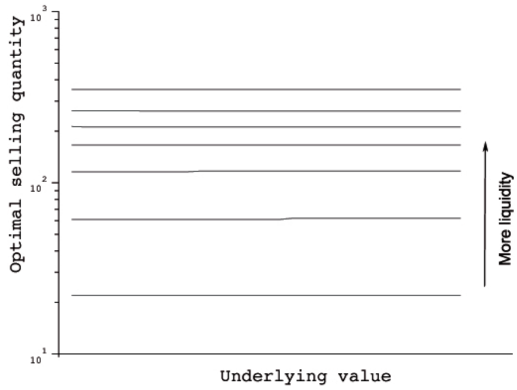

Figure 4.3: BS case: optimal selling quantities as function of the underlying value.

– The optimal strategy is almost linear with respect to the number of remaining stocks (see Figure 4.4). A slight concavity is equally worth noticing. This effect is explained by the presence of an exponential cost on optimization problem (4.3.14), which breaks the linearity of the problem and prevents very large orders.

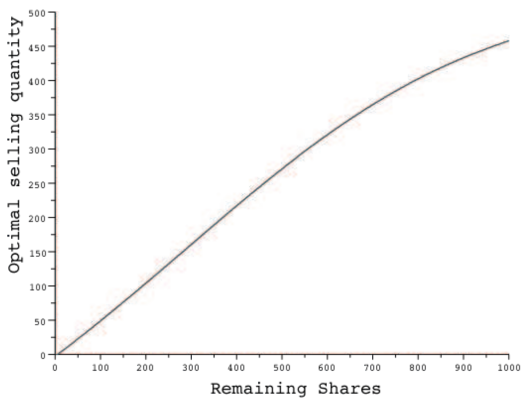

Figure 4.4: BS case: optimal selling quantities as function of the remaining shares owned by the investor.

– Finally, the main result is that the optimal strategy decreases when the Bid-Ask spread increases and the dependence is almost linear till the spread is lower than its long term average (see Figure 4.5). When the spread is bigger than its mean, the optimal strategy is to keep all remaining stocks. This result is very interesting since it says that the optimal strategy depends mainly on the Bid-Ask spread and its equilibrium law. The optimal strategy can be resumed by we have to sell when the spread is small and to wait a better time when it is wide.

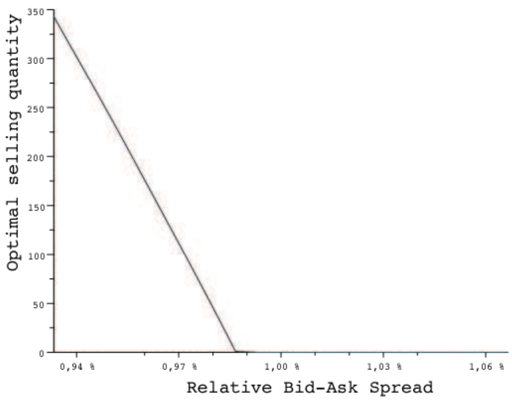

Figure 4.5: BS case: optimal selling quantities as function of the relative Bid-Ask spread.

4.4.2 CEV case

We consider the case when the underlying follows SDE (4.2.1) with $\alpha = -0.7$. The numerical simulations show that the results found in the Black-Scholes case remain true, except for the dependencies of the optimal selling strategies on underlying value.

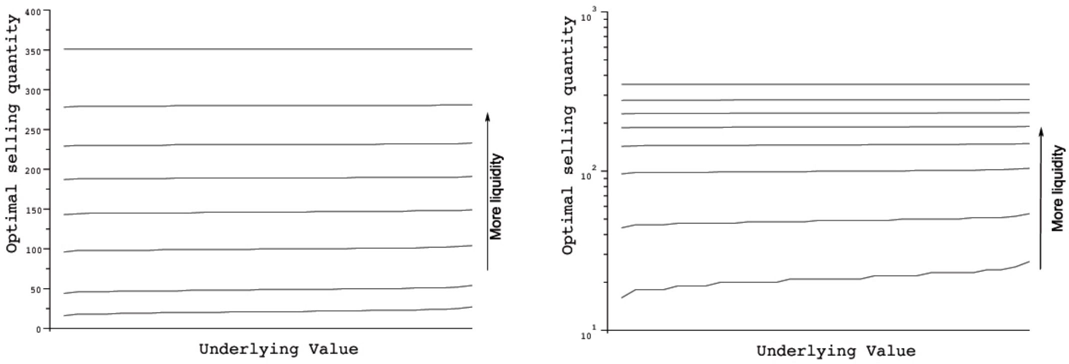

Figure 4.6: CEV case: optimal selling quantities as function of the relative Bid-Ask spread with linear scale (left) and logarithmic scale (right).

Remark 4.4.8. (CEV Case)

1. The optimal selling strategy is completely unaffected by underlying price when the stock is highly liquid. However, when it is illiquid, the optimal selling strategy is positively correlated with the price of underlying asset.

2. We also have analyzed the impact of a change on the CEV exponential α. When this parameter increases to zero, the dependency on the price of underlying asset is lessened.

Economic interpretation/explanation: We may explain the effect mentioned in the first point of Remark 4.4.8 by the non-linear dependency of the Bid-Ask spread on the underlying value (see Corollary 4.3.3). Indeed, when the price of underlying asset falls, the relative Bid-Ask spread (in percentage of the asset price) increases, which in turn incites the investor to delay the selling or sell a smaller number of shares.

The below Figure 4.7 shows the shape of the selling region (above the curve) and the non-selling region (below the curve) at a given time and a given level of stock price.

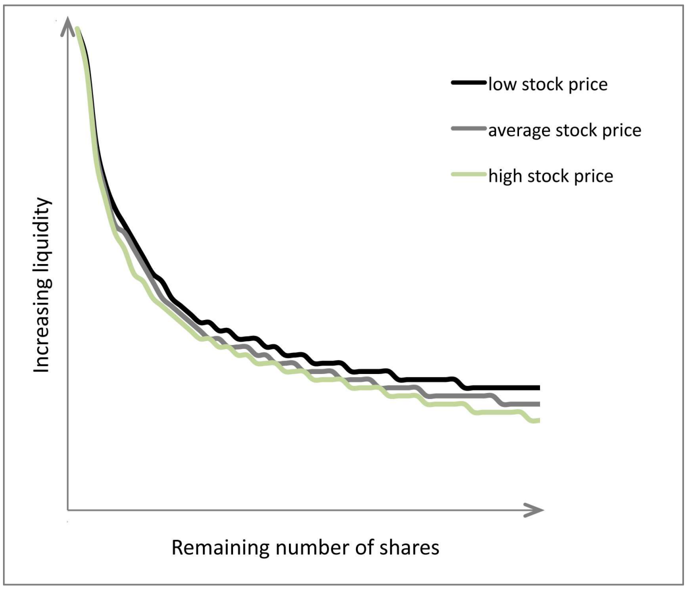

Figure 4.7: CEV case: the optimal strategy and the selling region and non-selling region.

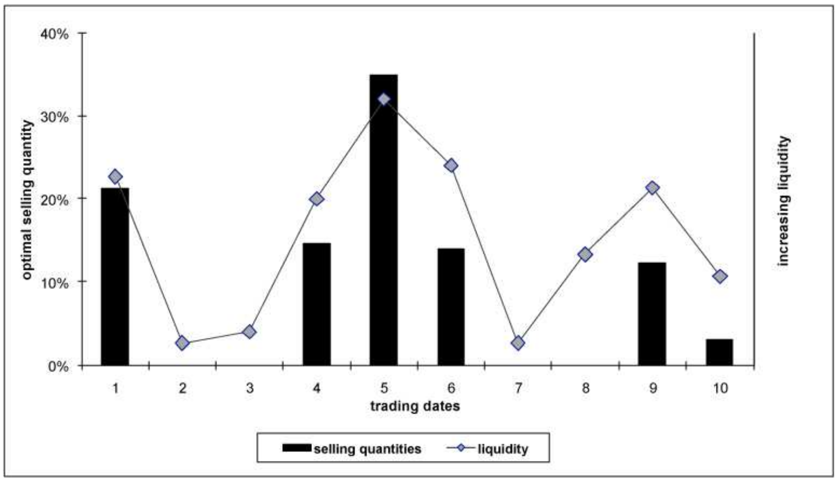

Figure 4.8: CEV case: an example of the optimal selling strategy given a liquidity trajectory.

4.4.3 Comparison on scenarios

In this section, we analyze the over performance of the optimal strategy with respect to the classical "one over n" one. To do that we compute the two gains obtained following the two previous strategies in the following scenarios:

average constant spread the Bid-Ask spread takes the mean value at all trading time;

increasing spread the Bid-Ask spread starts very tight, increases and ends very wide;

up-down spread the Bid-Ask spread takes tight values at odd times and wide at even ones;

random spread the Bid-Ask spread evolution is fixed randomly in accord with its law.

We treat only the CEV case, since the results for the Black-Scholes case are similar. The computation are performed under a scenario of constant underlying to neutralize its impact on the computation of extra returns. The following table resumes the optimal quantity to sell at each time.

strategies	1	2	3	4	5	6	7	8	9	10
Average spread	4.0	5.7	7.7	9.8	11.9	13.4	13.6	12.6	11.5	9.8
Increasing spread	40.8	22.0	9.0	1.8	0.0	0.0	0.0	0.0	15.0	11.4
Up & down	40.8	0.0	30.9	0.0	16.4	0.0	9.9	0.0	2.0	0.0
Random 1	21.2	0.0	0.0	14.6	34.9	14.0	0.0	0.0	12.3	3.0
Random 2	4.0	0.0	42.8	0.0	0.0	15.4	6.0	0.0	17.2	14.6
Random 3	0.0	16.9	39.6	0.0	7.5	19.4	0.0	3.1	8.0	5.5
Random4	0.0	22.8	0.0	33.9	0.0	25.4	12.9	0.0	5.0	0.0

Figure 4.9: Optimal strategies depending on spreads scenarios.

The relative extra profit, i.e. the ratio between the total gain obtained following the optimal strategy minus the one obtained following the "1 over n" one and the latter, is showed in the following table. We remark that the optimal strategy over perform the classical "1 over n" of around 0.7% in average. The worst performance is given by a non evolution of the spread that is unrealistic in our model. The best performance is obtained under a "roller coast" spread evolution, that is also unrealistic. In the four cases when the evolution is defined by using a Monte-Carlo approach, the optimal strategy outperform the classical "1 over n".

Strategies	A S	I S	U D	R 1	R 2	R 3	R 4
extra profit	−0.4%	0.2%	1.7%	0.9%	0.4%	0.8%	1.1%

Figure 4.10: Extra-return of optimal strategies depending on the evaluated scenarios.

4.5 Appendix

4.5.1 Proofs of Lemmas 4.2.1 and 4.2.2

We start with the proof of Lemma 4.2.1.

Proof. The proof is split into three steps.

Step 1: We compute the SDE satisfied by the sharp of S_t, see Section V.2 in Bouleau [25] for the definition and more details

$$dS_t^{\#} = r\, S_t^{\#}\, dt + \zeta(t,\, X_t,\, \omega)\, S_t^{\#}\, dW_t + \sqrt{\theta}\, \sigma(t,\, X_t,\, \omega)\, X_t\, d\tilde{W}_t,$$

where $\tilde{W}$ is an independent Brownian motion defined in a probability space $(\tilde{\Omega},\, \tilde{\mathcal{F}},\, \tilde{\mathbb{P}})$ copy of the original probability space. Assumption 4.2.1 insures that the previous SDE admits a square integrable solution.

Step 2: We apply Itô formula to $(S_t^{\#})^2$ and take the expectation under the probability $\tilde{\mathbb{P}}$, since one of the properties of the sharp operator is that $\Gamma[S_t] = \tilde{\mathbb{E}}[(S_t^{\#})^2]$, see Sections VI.2 and VII.4 in Bouleau [25]. Therefore we find SDE (4.2.6).

Step 3: Finally, we prove that SDE (4.2.6) admits the closed form solution (4.2.5). Using the methods developed by Bouleau [26] Section 5 and noticing that the SDE verified by the sharp is linear, we may apply a variation of constant method, see for instance Protter [115] Section V.9, and obtain a closed form for $S_t^{\#}$. Then, we easily compute the expectation under $\tilde{\mathbb{P}}$ of the square of $S_t^{\#}$. Another possibility is to check that first equation in (4.2.5) is solution to SDE (4.2.6). □

We now turn to the proof of Lemma 4.2.2.

Proof. The proof is based on a L^2-convergence argument, see for instance Da Prato [43], by using the fact that operators $\Gamma[\cdot]$ and $\mathcal{A}[\cdot]$ are closed, see Bouleau [25].

We define a partition $\{\tau_i\}_{i=1,\dots,n}$ of the interval $[0,\, T]$, where T is a sufficient large time. We approximate X with the following process

$$Z_t = \sum_{i=1}^{n} \sigma(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, Z_{\tau_{i-1}}\, (W_{\tau_i \wedge t} - W_{\tau_{i-1}\wedge t}) + r \sum_{i=1}^{n} Z_{\tau_{i-1}}\, (\tau_i \wedge t - \tau_{i-1} \wedge t).$$

It is clear that Z_t converges to X_t when the partition step goes to zero, thanks to hypotheses 1 and 3 of Assumption 4.2.1. Then, we apply the bias operator on Z_t (see Section 6 in [26]), and we find

$$\begin{aligned}
\mathcal{A}[Z_t] = & \sum_{i=1}^{n} \zeta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, \mathcal{A}[Z_{\tau_{i-1}}]\, (W_{\tau_i \wedge t} - W_{\tau_{i-1}\wedge t}) \\
& + r \sum_{i=1}^{n} \mathcal{A}[Z_{\tau_{i-1}}]\, (\tau_i \wedge t - \tau_{i-1} \wedge t) \\
& + \sum_{i=1}^{n} \sigma(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega) Z_{\tau_{i-1}}\, \mathcal{A}[W_{\tau_i \wedge t} - W_{\tau_{i-1}\wedge t}] \\
& + \sum_{i=1}^{n} \zeta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, \tilde{\mathbb{E}}\big[\big(Z_{\tau_{i-1}}^{\#}\big)\, \big(W_{\tau_i \wedge t}^{\#} - W_{\tau_{i-1}\wedge t}^{\#}\big)\big] \\
& + \frac{1}{2} \sum_{i=1}^{n} \eta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, Z_{\tau_{i-1}}\, \Gamma[Z_{\tau_{i-1}}]\, (W_{\tau_i \wedge t} - W_{\tau_{i-1}\wedge t})\,.
\end{aligned} \tag{4.5.21}$$

Then, we take the limit and SDE (4.5.21), when the partition step goes to zero and we have to prove that equation (4.5.21) converges to the integral form of equation (4.2.7). For sake of simplicity on notation, we compute all equations at the final time T. We start with the remark that Z_t converges to X_t in L^2-norm when the partition step goes to zero. It's also clear that $\Gamma[Z_t]$ converges to $\Gamma[S_t]$ due to the fact that Γ is a closed operator, see for instance [24], and the result of Lemma 4.2.1, this convergence is in L^1-norm but it is easy to check that it is true in L^2-norm too. Then, under Assumption 4.2.1, we will prove that

$$\sum_{i=1}^{n} \eta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\,\Gamma[Z_{\tau_{i-1}}]\,(W_{\tau_i\wedge t} - W_{\tau_{i-1}\wedge t}) \xrightarrow{L^2} \int_0^T \eta(t,\, X_t,\, \omega)\,\Gamma[S_t]\,dW_t\,. \qquad (4.5.22)$$

We separate the last integral using the partition $(\tau_i)_{i=1,\ldots,n}$ and we evaluate the difference in L^2-norm, so we find

$$\begin{aligned}
&\mathbb{E}\Big[\Big\{\,\eta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\,\Gamma[Z_{\tau_{i-1}}](W_{\tau_i} - W_{\tau_{i-1}}) - \int_{\tau_{i-1}}^{\tau_i} \eta(t,\, X_t,\, \omega)\,\Gamma[S_t]\,dW_t\,\Big\}^2\Big]\\
&= \mathbb{E}\Big[\Big\{\eta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\,\Gamma[Z_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}}) - \eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\,\Gamma[Z_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}})\\
&\quad + \eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\,\Gamma[Z_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}}) - \eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\,\Gamma[S_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}})\\
&\quad + \eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\,\Gamma[S_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}}) - \int_{\tau_{i-1}}^{\tau_i} \eta(t,\, X_t,\, \omega)\,\Gamma[S_t]\,dW_t\Big\}^2\Big]\\
&< \mathbb{E}\big[\big\{[\eta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega) - \eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)]\,\Gamma[Z_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}})\big\}^2\big] +\\
&\quad + \mathbb{E}\big[\big\{\eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\,[\Gamma[Z_{\tau_{i-1}}] - \Gamma[S_{\tau_{i-1}}]]\,(W_{\tau_i} - W_{\tau_{i-1}})\big\}^2\big] +\\
&\quad + \mathbb{E}\Big[\Big\{\eta(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\,\Gamma[S_{\tau_{i-1}}]\,(W_{\tau_i} - W_{\tau_{i-1}}) - \int_{\tau_{i-1}}^{\tau_i} \eta(t,\, X_t,\, \omega)\,\Gamma[S_t]dW_t\Big\}^2\Big].
\end{aligned}$$

The first expectation converges to zero thanks to the Lipschitz hypothesis on $\eta(t,\, x,\, \omega)$ w.r.t x. The second expectation goes to zero using the fact that $\Gamma[Z_t]$ converges to $\Gamma[S_t]$ in L^2-norm. The last expectation converges to zero when the partition step goes to zero in accordance with the definition of the stochastic integral. Then, the limit (4.5.22) is proved.

Using the same arguments and Gronwall lemma, see for instance Protter [115] Chapter V, we have

$$\sum_{i=1}^{n} \zeta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\,\mathcal{A}[Z_{\tau_{i-1}}]\,(W_{\tau_i\wedge t} - W_{\tau_{i-1}\wedge t}) \xrightarrow{L^2} \int_0^T \zeta(t,\, X_t,\, \omega)\,\mathcal{A}[S_t]\,dW_t\,, \qquad (4.5.23)$$

and

$$\sum_{i=1}^{n} \mathcal{A}[Z_{\tau_{i-1}}]\,(\tau_i - \tau_{i-1}) \xrightarrow{L^2} \int_0^T \mathcal{A}[S_t]\,dt\,. \qquad (4.5.24)$$

We study the third term in equation (4.5.21). We find

$$\sum_{i=1}^{n} \sigma(\tau_{i-1}, Z_{\tau_{i-1}}, \omega) Z_{\tau_{i-1}} \mathcal{A}[W_{\tau_i \wedge t} - W_{\tau_{i-1} \wedge t}] = -\frac{\theta}{2} \sum_{i=1}^{n} \sigma(\tau_{i-1}, Z_{\tau_{i-1}}, \omega) Z_{\tau_{i-1}} [W_{\tau_i} - W_{\tau_{i-1}}],$$

thanks to the chain rule of semigroup $\mathcal{A}$, see Section 3 in [26]. We also remark that $\sigma(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, Z_{\tau_{i-1}}$ converges to $\sigma(\tau_{i-1},\, X_{\tau_{i-1}},\, \omega)\, X_{\tau_{i-1}}$, thanks to Assumption 4.2.1. Using always the same arguments used to prove limit (4.5.22), we have therefore that

$$\sum_{i=1}^{n} \sigma(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, Z_{\tau_{i-1}}\, \mathcal{A}[W_{\tau_i \wedge t} - W_{\tau_{i-1} \wedge t}] \xrightarrow{L^2} -\frac{\theta}{2} \int_0^T \sigma(t,\, X_t,\, \omega)\, X_t \, dW_t \,. \qquad (4.5.25)$$

Finally, we analyze the term

$$\sum_{i=1}^{n} \zeta(\tau_{i-1},\, Z_{\tau_{i-1}},\, \omega)\, \tilde{\mathbb{E}}\big[\big(Z^{\#}_{\tau_{i-1}}\big)\, \big(W^{\#}_{\tau_i \wedge t} - W^{\#}_{\tau_{i-1} \wedge t}\big)\big].$$

We introduce a conditional expectation with respect to the σ-algebra $\tilde{\mathcal{F}}_{\tau_{i-1}} = \sigma\{W^{\#}_s, W_u |\, u,\, s < \tau_{i-1}\}$

$$\begin{aligned}
\sum_{i=1}^{n} \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[\Big(Z^{\#}_{\tau_{i-1}}\Big)\Big(W^{\#}_{\tau_i \wedge t} - W^{\#}_{\tau_{i-1} \wedge t}\Big)\,\Big|\, \tilde{\mathcal{F}}_{\tau_{i-1}}\Big]\Big] &= \sum_{i=1}^{n} \tilde{\mathbb{E}}\,\Big[\Big(Z^{\#}_{\tau_{i-1}}\Big)\, \tilde{\mathbb{E}}\Big[W^{\#}_{\tau_i} - W^{\#}_{\tau_{i-1}}\,\Big|\, \tilde{\mathcal{F}}_{\tau_{i-1}}\Big]\Big] \\
&= 0\,,
\end{aligned}$$

using the fact that $Z^{\#}_s$ is adapted to the filtration $\tilde{\mathcal{F}}_s$ and $W^{\#}_s$ remains a Brownian motion w.r.t filtration $\tilde{\mathcal{F}}_s$, since $W^{\#}_s$ and W_s are independent. As a consequence, the fourth term in equation (4.5.21) is always equal to zero and we have proved the convergence of equation (4.5.21) to the integral form of equation (4.2.7). Now it is easy to check that second equation in (4.2.5) solves SDE (4.2.7).

$\square$

Bibliography

[1] Alfonsi, A., A. Schied and A. Schulz (2007): "Optimal execution strategies in limit order books with general shape functions", preprint.

[2] Almgren, R. and N. Chriss (2000): "Optimal execution of portfolio transaction", *Journal of Risk*, **3**, 5-39.

[3] Ankirchner, S., C. Blanchet-Scalliet and A. Eyraud-Loisel (2009): "Credit risk premia and quadratic BSDEs with a single jump", preprint.

[4] Bain, A. and D. Crisan (2009): "Fundamentals of stochastic filtering", Springer.

[5] Bank, P. and D. Baum (2004): "Hedging and portfolio optimization in illiquid financial markets with a large trader", *Mathematical Finance*, **14**, 1-18.

[6] Bardou, O., S. Bouthemy and G. Pages (2009): "Optimal quantization for the pricing of swing options", *Applied Mathematical Finance*, **16**, 183-217.

[7] Barles, G., R. Buckdahn and E. Pardoux (1997): "Backward stochastic differential equations and integral-partial differential equations ", *Stochastics and Stochastics Reports*, **60**, 57-83.

[8] Barrera-Esteve, C., F. Bergeret, C. Dossal, E. Gobet, A. Meziou, R. Munos and D. Reboul-Salze (2006): "Numerical methods for the pricing of swing options: a stochastic control approach", *Methodology and Computing in Applied Probability*, **8**, 104-128.

[9] Bellamy, N. (2001): "Wealth optimization in an incomplete market driven by a jump-diffusion process", *Journal of Mathematical Economics*, **35**, 259-287.

[10] Bertsekas, D. and E. Shreve (1978): "Stochastic optimal control: the discrete time case", *Mathematics in Science and Engineering*, **139**.

[11] Bertsimas, D. and A. Lo (1998): "Optimal control of execution costs", *Journal of Financial Markets*, **1**, 1-50.

[12] Biagini, S., M. Frittelli and M. Grasselli (2008): "Indifference price with general semimartingales", preprint.

[13] Bialkowski, D., S. Darolles and G. Le Fol (2008): "Improving VWAP strategies: a dynamic volume approach", *Journal of Banking and Finance*, **32**, 1709-1722.

[14] Bielecki, T., M. Jeanblanc and M. Rutkowski (2004): "Hedging of defaultable claims", Lectures notes in mathematical finance, Springer.

[15] Bielecki, T., M. Jeanblanc and M. Rutkowski (2004): "Stochastic methods in credit risk modelling", Lectures notes in Mathematics, Springer, **1856**, 27-128.

[16] Bielecki, T. and M. Rutkowski (2004): "Credit risk: modeling, valuation and hedging", Springer Finance.

[17] Bielecki, T. and M. Jeanblanc (2008): "Indifference prices in indifference pricing", *Theory and Applications, Financial Engineering*, Princeton University Press. R. Carmona editor volume.

[18] Bismut, J.M. (1976): "Théorie probabiliste du contrôle des diffusions", *Mem. Aler. Math. Soc.*, **4**.

[19] Black, F. (1976): "Studies of stock price volatility changes", *Pr. Meeting American Stat. Association, Business and Economics Statistic Division*, 177-181.

[20] Blanchet-Scalliet, C. and M. Jeanblanc (2004): "Hazard rate for credit risk and hedging defaultable contingent claims", *Finance and Stochastics*, **8**, 145-159.

[21] Bouchard, B. (2006): "No-arbitrage in discrete-time markets with proportional transaction costs and general information structure", *Finance and Stochastics*, **10**, 276-297.

[22] Bouchard, B. and R. Elie (2008): "Discrete-time approximation of decoupled Forward-Backward SDE with jumps", *Stochastic Processes and their Applications*, **118**, 53-75.

[23] Bouchaud, J.P., M. Mézard and M. Potters (2002): "Statistical properties of stock order books: empirical results and models", preprint.

[24] Bouleau, N. and F. Hirsch (1991): "Dirichlet forms and analysis on Wiener space", De Gruyter, Berlin.

[25] Bouleau, N. (2003): "Error calculus for finance and physics", De Gruyter, Berlin.

[26] Bouleau, N. (2003): "Error calculus and path sensivity in financial models", *Mathematical Finance*, **13**, 115-134.

[27] Bouleau, N. (2006): "When and how an error yields a Dirichlet form", *Journal of Functional Analysis*, **240**, 445-494.

[28] Briand, P. and Y. Hu (2006): "BSDE with quadratic growth and unbounded terminal value", *Probability Theory and Related Fields*, **136**, 604-618.

[29] Briand, P., J.P. Lepeltier and J. San Martin (2007): "One-dimensional backward stochastic differential equations whose coefficient is monotonic in y and non-Lipschitz in z", *Bernoulli*, **13**, 80-91.

[30] Brunnermeie, M. and L.H. Pedersen (2008): "Market liquidity and funding liquidity", *Review of Financial Studies*.

[31] Brunnermeie, M. (2009): "Deciphering the liquidity and credit crunch 2007-2008", *Journal of Economic Perspectives*, **23**, 77-100.

[32] Callegaro, G., G. Di Masi and W. Runggaldier (2006): "Portfolio optimization in discontinuous markets under incomplete information", *Asia-Pacific Financial Markets*, **13**, 373-394.

[33] Carmona, R. (2008): "Volume on indifference pricing", Princeton University Press.

[34] Cetin, U., R. Jarrow and P. Protter (2004): "Liquidity risk and arbitrage pricing theory", *Finance and Stochastics*, **8**, 311-341.

[35] Cetin, U. and L.C.G. Rogers (2007): "Modelling liquidity effects in discrete time", *Mathematical Finance*, **17**, 15-29.

[36] Cetin, U., M. Soner and N. Touzi (2007): "Option hedging under liquidity costs", *Finance and Stochastics*, to appear.

[37] Cont, R., S. Stoikov and R. Talreja (2008): "A stochastic model for order book dynamics", *Operations Research*, to appear.

[38] Corsi, M., H. Pham and W. Runggaldier (2008): "Numerical approximation by quantization of control problems in finance under partial observations", *Mathematical modelling and numerical methods in finance*, Handbook of Numerical Analysis, **15**, 325-360.

[39] Cox, J. (1975): "Notes on option pricing I: Constant elasticity of diffusions", Stanford University.

[40] Cox, J. and S. Ross (1976): "The valuation of options for alternative stochastic processes", *Journal of Financial Economics*, **3**, 145-166.

[41] Cox, J. and C. Huang (1989): "Optimal consumption and portfolio policies when asset prices follow a diffusion process", *Journal of Economic Theory*, **49**, 33-83.

[42] Cuoco, D. and J. Cvitanic (1998): "Optimal consumption choice for a large investor", *Journal of Economic Dynamics and Control*, **22**, 401-436.

[43] Da Prato, G. (2006): "An introduction to infinite-dimensional analysis", Springer, Berlin.

[44] Darling, R. and E. Pardoux (1997): "Backwards SDE with random terminal time and applications to semilinear elliptic PDE", *The Annals of Probability*, **25**, 1135-1159.

[45] Delbaen, F., P. Grandits, T. Rheinlander, D. Samperi, M. Schweizer and C. Stricker (2002): "Exponential hedging and entropic penalties", *Mathematical Finance*, **12**, 99-123.

[46] Dellacherie, C. and P.A. Meyer (1980): "Probabilités et potentiel", Théorie des martingales, Hermann.

[47] Detemple, J. (1986): "Asset pricing in a production economy with incomplete information", *Journal of Finance*, **41**, 383-391.

[48] Dothan, M.U. and D. Feldman (1986): "Equilibrum interest rates and multiperiod bonds in a partially observable economy", *Journal of Finance*, **41**, 369-382.

[49] Duffie, D. and L. Epstein (1992): "Stochastic differential utility", *Econometrica*, **60**, 353-394.

[50] Duffie, D. and K. Singleton (2003): "Credit risk: pricing, measurement and management", Princeton University Press.

[51] Dupire, B. (1994): "Pricing with a smile", *Risk Magazine*.

[52] Ekeland, I. and R. Temam (1976): "Convex analysis and variational problems", North-Holland-Elsevier.

[53] El Karoui, N. (1979): "Les aspects probabilistes du contrôle stochastique", Ecole d'Eté de Probabilités de Saint-Flour IX, Springer-Verlag.

[54] El Karoui, N. and M.C. Quenez (1995): "Dynamic programming and pricing of contingent claims in an incomplete market", *SIAM Journal on Control and Optimization*, **33**, 29-66.

[55] El Karoui, N. and L. Mazliak (1997): "Backward stochastic differential equations", Pitman research notes in mathematical series.

[56] El Karoui, N., S. Peng and M.C. Quenez (1997): "Backward stochastic differential equations in finance", *Mathematical finance*, **7**, 1-71.

[57] El Karoui, N. and M.C. Quenez (1997): "Imperfect markets and backward stochastic differential equations", *Numerical methods in finance*, 181-214.

[58] El Karoui, N., M. Jeanblanc, A. Matoussi and A. Ngoupeyou (2010): "Quadratic Backward SDE's with jumps and utility maximization of portfolio credit derivatives', *Preprint*.

[59] Elliott, M., M. Jeanblanc and M. Yor (2000): "On models of default risk", *Mathematical Finance*, **10**, 179-195.

[60] Frey, R. (1998): "Perfect option hedging for a large trader", *Finance and Stochastics*, **2**, 115-141.

[61] Frey, R. and W. Runggaldier (1999): "Risk-minimizing hedging strategies under restricted information: the case of stochastic volatility models observable only at discrete random times", *Finance and Stochastics*, to appear.

[62] Frino, A. and D. Gallagher (2001): "Tracking S&P 500 Index Funds", *Journal Portfolio Management*, **28**, 44-55.

[63] Gennotte, G. (1986): "Optimal portfolio choice under incomplete information", *Journal of Finance*, **41**, 733-746.

[64] Hamadène, S. and J.P. Lepeltier (1995): "Backward equations, stochastic control and zero-sum stochastic differential games", *Stochastics and stochastic reports*, **54**, 221-231.

[65] He, H. and H. Mamaysky (2005): "Dynamic trading policies with price impact", *Journal of Economic Dynamics and Control*, **29**, 891-930.

[66] Hodges, S.D. and A. Neuberger (1989): "Optimal replication of contingent claim under transaction costs", *Review of Futures Markets*, **8**, 222-239.

[67] Hu, Y., P. Imkeller and M. Muller (2005): "Utility maximization in incomplete markets", *The Annals of Applied Probability*, **15**, 1691-1712.

[68] Hull, J. and A. White (1987): "The pricing of options on assets with stochastic volatilities", *Journal of Finance*, **42**, 28-300.

[69] Jacod, J. (1987): "Grossissement initial, hypothèse H' et théorème de Girsanov, Séminaire de calcul stochastique", Lectures Notes in Maths, vol. 1118.

[70] Jarrow, R.A. and F. Yu (2001): "Counterparty risk and the pricing of defaultable securities", *Journal of Finance*, **56**, 1765-1799.

[71] Jeanblanc, M. and M. Pontier (1990): "Optimal portfolio for a small investor in a market with discontinuous prices", *Applied mathematics and optimization*, **22**, 287-310.

[72] Jeanblanc, M. and Y. Le Cam (2009): "Progressive enlargement of filtrations with initial times", *Stochastic Processes and their Applications*, to appear.

[73] Jeanblanc, M., M. Yor and M. Chesney (2009): "Mathematical methods for financial markets", Springer Finance.

[74] Jeulin, T. (1980): "Semimartingales et grossissements d'une filtration", Lectures Notes in Maths, vol. 883, Springer.

[75] Jeulin, T. and M. Yor (1985): "Grossissement de filtration : exemples et applications", Lectures Notes in Maths, vol. 1118, Springer.

[76] Jiao, Y. and H. Pham (2009): "Optimal investment with counterparty risk: a default-density modeling approach", *Finance and Stochastics*, to appear.

[77] Karatzas, I., J.P. Lehoczky and S. Shreve (1987): "Optimal portfolio and consumption decisions for a small investor on a finite horizon", *SIAM Journal on Control and Optimization*, **25**, 1557-1586.

[78] Karatzas, I., J.P. Lehoczky, S. Shreve and G. Xu (1991): "Martingale and duality methods for utility maximization in an incomplete market", *SIAM Journal on Control and Optimization*, **29**, 702-730.

[79] Karatzas, I. and S. Shreve (1991): "Brownian motion and stochastic calculus", Springer Verlag.

[80] Karatzas, I. and X. Xue (1991): "A note on utility maximization under partial observations", *Mathematical Finance*, **1**, 57-70.

[81] Karatzas, I. and X. Zhao (1998): "Bayesian adaptative portfolio optimization", preprint.

[82] Kharroubi, I. and H. Pham (2009): "Optimal portfolio liquidation with execution cost and risk", preprint.

[83] Kobylanski, M. (2000): "Backward stochastic differential equations and partial differential equations with quadratic growth", *The Annals of Probability*, **28**, 558-602.

[84] Kramkov, D. and W. Schachermayer (1999): "The asymptotic elasticity of utility functions and optimal investment in incomplete markets", *The Annals of Applied Probability*, **9**, 904-950.

[85] Kusuoka, S. (1999): "A remark on default risk models", *Advances in Mathematical Economics*, **1**, 69-82.

[86] Lakner, P. (1995): "Utility maximization with partial information", *Stochastic Processes and their Applications*, **56**, 247-273.

[87] Lakner, P. (1998): "Optimal trading strategy for an investor: the case of partial information", *Stochastic Processes and their Applications*, **76**, 77-97.

[88] Lasry, J.M. and P.L. Lions (1999): "Stochastic control under partial information and applications to finance", *Stochastic Processes and their Applications*, **56**, 247-273.

[89] Lehalle, C.A. (2009): "The impact of liquidity fragmentation on optimal trading", *Trading*, **1**, 80-487.

[90] Lepeltier, J.P. and J. San Martin (1998): "Existence for BSDE with superlinear-quadratic coefficient", *Stochastics and stochastic reports*, **63**, 227-240.

[91] Lim, T. and M.C. Quenez (2009): "Utility maximization in incomplete market with default", preprint.

[92] Lim, T., V. Ly Vath and S. Scotti (2010): "Bid-Ask spread modelling, a perturbation approach", preprint.

[93] Lim, T. and M.C. Quenez (2010): "Portfolio optimization in a default model under full/partial information", preprint.

[94] Lukas, S. (2001): "On pricing and hedging defaultable contingent claims", Thesis, Humbolt University.

[95] Ly Vath, V., M. Mnif and H. Pham (2007): "A model of optimal portfolio selection under liquidity risk and price impact", *Finance and Stochastics*, **11**, 51-90.

[96] Macbeth, J. and L. Merville (1980): "Test of the Black Scholes and Cox call option valuation models", *Journal of Finance*, **35**, 285-301.

[97] Mania, M. and M. Schweizer (2005): "Dynamic exponential utility indifference valuation", *Annals of Applied Probability*, **15**, 2113-2143.

[98] Merton, R.C. (1971): "Optimum consumption and portfolio rules in a continuous-time model", *Journal of Economic Theory*, **3**, 373-413.

[99] Morlais, M.A. (2009): "Utility maximization in a jump market model", *Stochastics and Stochastics Reports*, **81**, 1-27.

[100] Neveu, J. (1975): "Discrete-parameter martingales", English translation, North-Holland, Amsterdam and American Elsevier, New York.

[101] Obizhaeva, A. and J. Wang (2005): "Optimal trading strategy and supply/demand dynamics", preprint.

[102] Pardoux, E. (1989): "Filtrage non linéaire et équations aux dérivées partielles stochastiques associées", Lectures Notes in Math., Springer Verlag, 1464, 67-163.

[103] Pardoux, E. and S. Peng (1990): "Adapted solution of a backward stochastic differential equation", *Systems Control Lett.*, **14**, 55-61.

[104] Pardoux, E. and S. Peng (1992): "Backward stochastic differential equations and quasilinear parabolic partial differential equations", Lecture Notes in Control and Inform. Sci., **176**, 200-217.

[105] Pardoux, E. (1997): "Generalized discontinuous backward stochastic differential equation", in: N, El Karoui, L. Mazliak (Eds.), Backward Stochastic Differential Equations, in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 207-219.

[106] Parlour, C.A. (1998): "Price dynamics in limit order markets", *Review of Financial Studies*, **11**, 789-816.

[107] Peng, S. (1990): "A general stochastic maximum principle for optimal control problems", *SIAM Journal on control and optimization*, **28**, 966-979.

[108] Peng, S. (1991): "Probabilistic interpretation for systems of quasilinear parabolic partial differential equations", *Stochastics and Stochastics Reports*, **37**, 61-74.

[109] Peng, S. and X. Xu (2009): "BSDEs with random default time and their applications to default risk", preprint.

[110] Pham, H. and M.C. Quenez (2001): "Optimal portfolio in partially observed stochastic volatility models", *Annals of Applied Probability*, **11**, 210-238.

[111] Pham, H. (2009): "Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management", preprint.

[112] Platen, E. and M. Schweizer (1998): "On feedback effects from hedging derivatives", *Mathematical Finance*, **8**, 67-84.

[113] Platen, E. and W. Runggaldier (2007): "A benchmark approach to portfolio optimization under partial information", *Asia Pacific Financial Markets*, **14**, 25-43.

[114] Potters, M. and J.P. Bouchaud (2003): "More statistical properties of order books and price impact", *Physica A*, **243**, 133-140.

[115] Protter, P. (1990): "Stochastic integration and differential equations", Springer Verlag.

[116] Roland, S. (2007): "Partially observed jump diffusions", Thesis.

[117] Rouge, R. and N. El Karoui (2000): "Pricing via utility maximization and entropy", *Mathematical Finance*, **10**, 259-276.

[118] Royer, M. (2006): "Backward stochastic differential equations with jumps and related non-linear expectations", *Stochastic processes and their applications*, **116**, 1358-1376.

[119] Runggaldier, W. (2010): "Filtering", *Encyclopedia of quantitative finance*, 674-683.

[120] Saas, J. and U. Haussmann (2004): "Optimizing the terminal wealth under partial information: the drift process as a continuous Markov chain", *Finance and Stochastics*, **8**, 553-577.

[121] Schachermayer, W. (2001): "Optimal investment in incomplete markets when wealth may become negative", *Annals of Applied Probability*, **11**, 694-734.

[122] Schied, A. and T. Schoneborn (2009): "Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets", *Finance and Stochastics*, **13**, 181-204.

[123] Schonbucher, P.J. and P. Wilmott (2000): "The feedback effect of hedging in illiquid markets", *SIAM Journal on Applied Mathematics*, **61**, 232-272.

[124] Sekine, J. (2006): "On exponential hedging and related quadratic backward stochastic differential equations ", *Applied Mathematics and Optimization*, **54**, 131-158.

[125] Stricker, C. (1983): "Quelques remarques sur les semi-martingales gaussiennes et le problème de l'innovation", Lectures Notes in Cont. and Inf. Sci., Springer Verlag, **61**, 260-276.

[126] Tang, S.H. and X. Li (1994): "Necessary conditions for optimal control of stochastic systems with random jumps", *SIAM Journal on Control and Optimization*, **32**, 1447-1475.

[127] Wang, H.K. and J. Yau (2000): "Trading volume; bid-ask spread, and price volatility in futures markets", *Journal of Futures markets*, **20**, 943-970.

Résumé : Cette thèse se compose de trois parties indépendantes portant sur l'application du contrôle stochastique à la finance. Dans la première partie, nous étudions le problème de maximisation de la fonction d'utilité dans un marché incomplet avec défauts et information totale/partielle. Nous utilisons le principe de la programmation dynamique pour pouvoir caractériser la fonction valeur solution du problème. Ensuite, nous utilisons cette caractérisation pour en déduire une EDSR dont la fonction valeur est solution. Nous donnons également une approximation de cette fonction valeur. Dans la seconde partie, nous étudions les EDSR à sauts. En utilisant les résultats de décomposition des processus à sauts liée au grossissement progressif de filtration, nous faisons un lien entre les EDSR à sauts et les EDSR browniennes. Cela nous permet de donner un résultat d'existence, un théorème de comparaison ainsi qu'une décomposition de la formule de Feynman-Kac. Puis nous utilisons ces techniques pour la détermination du prix d'une option européenne dans un marché complet et le prix d'indifférence d'un actif contingent non duplicable dans un marché incomplet. Enfin, dans la troisième partie, nous utilisons la théorie des erreurs pour expliquer le risque de liquidité comme une propriété intrinsèque au marché. Cela nous permet de modéliser la fourchette *Bid-Ask*. Puis nous résolvons dans ce modèle le problème de liquidation optimale d'un portefeuille en temps discret et déterministe en utilisant la programmation dynamique.

Mots-clés : Maximisation d'utilité, temps de défaut, programmation dynamique, EDSR, prix d'indifférence, information partielle, grossissement progressif de filtration, décomposition dans la filtration de référence, théorème de comparaison, formule de Feynman-Kac, risque de liquidité, théorie des erreurs.

Abstract : This PhD dissertation consists of three independent parts and deals with applications of stochastic control to finance. In the first part, we study the utility maximization problem in a market with defaults and total/partial information. The dynamic programming principle is used to characterize the value function. Given this characterization, we find a BSDE of which the value function is a solution. We also give an approximation of this value function. In the second part, we study BSDEs with jumps. We link BSDEs with jumps and Brownian BSDEs using the decomposition of processes in the reference filtration. With this link, we get a result of existence, a comparison theorem and a decomposition of Feynman-Kac formula. We use these techniques to work out the price of a European option in a complete market and the indifference price of a contingent claim in an incomplete market. Finally, in the third part, we use the error theory to explain the liquidity risk and to model the Bid-Ask spread. Then we solve an optimal liquidation problem for a large portfolio in discrete and deterministic time.

Key words : Utility maximization, default times, dynamic programming, BSDE, indifference pricing, partial information, progressive enlargement of filtrations, decomposition in the reference filtration, Feynman-Kac formula, liquidity risky, error theory.

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